ON FULLY NONLINEAR CR INVARIANT EQUATIONS ON THE HEISENBERG GROUP

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Abstract. In this paper we provide a characterization of second order fully nonlinear CR invariant equations on the Heisenberg group, which is the analogue in the CR setting of the result proved in the Euclidean setting by A. Li and the first author in [21]. We also prove a comparison principle for solutions of second order fully nonlinear CR invariant equations defined on bounded domains of the Heisenberg group and a comparison principle for solutions of a family of second order fully nonlinear equations on a punctured ball.

1. Introduction and main results

As it was pointed out by Jerison and Lee in [16], there are important similarities between conformal geometry and the geometry of CR manifolds, which serve as abstract models of real hypersurfaces in complex manifolds, which we are going to discuss briefly in the following. On this subject see also the survey article by Beals, Fefferman and Grossman [2] and the book by Dragomir and Tomassini [8].

A CR manifold is a differentiable manifold M equipped with a subbundle $\mathcal{H}$ of the complexified tangent bundle $\mathbf{C}TM = TM \otimes \mathbf{C}$ such that $[\mathcal{H}, \mathcal{H}] \subseteq \mathcal{H}$ (i.e. $\mathcal{H}$ is formally integrable) and $\mathcal{H} \cap \bar{\mathcal{H}} = \{0\}$ (i.e. $\mathcal{H}$ is almost Lagrangian). The bundle $\mathcal{H}$ is called a CR structure on the manifold M. An abstract CR manifold is said to be of hypersurface type if $\dim_{\mathbf{R}} M = 2n + 1$ and $\dim_{\mathbf{C}} \mathcal{H} = n$.

If the CR manifold M is of hypersurface type and oriented, it is possible to associate to its CR structure $\mathcal{H}$ a one form θ globally defined on M such that $Ker(\theta) = \mathcal{H}$. θ is unique modulo a multiple of nonzero function on M: a choice of a nonzero multiple of such θ is called a pseudohermitian structure on M. $d\theta$ defines the Levi form L_θ on $\mathcal{H}$. If the Levi form L_θ is strictly positive definite, we say that M, with this CR structure, is strictly pseudoconvex. In this case, the form θ, as well as a nonzero function multiple of θ, is a contact form on M. See for instance [8].

A scalar curvature associated to the pseudohermitian structure has been introduced by Webster in [30], [31] and by Tanaka in [27]. Thus the CR Yamabe problem is formulated as follows: on a strictly pseudoconvex CR manifold, find a choice of pseudohermitian structure, or equivalently a choice of contact form θ, with constant pseudohermitian scalar curvature.

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Jerison and Lee proved in [16] that there exists a CR numerical invariant $\lambda(M)$ associated to every compact strictly pseudoconvex orientable CR manifold M of dimension $2n+1$, such that $\lambda(M)$ is always less than or equal to the value corresponding to the sphere $S^{2n+1} \subset \mathbf{C}^{n+1}$ and such that the CR Yamabe problem admits solution on M, provided that $\lambda(M) < \lambda(S^{2n+1})$. This result is an analogue in the CR setting of the classical result of Aubin [1] on Riemannian manifolds.

Jerison and Lee also proved in [17] that if θ is a contact form associated to the standard CR structure on S^{2n+1} having constant pseudohermitian scalar curvature, then it is obtained from a constant multiple of the standard contact form on S^{2n+1} via a CR automorphism of the sphere. This result is then an analogue in the CR setting of the well known result by Obata in [26] and by Gidas, Ni and Nirenberg in [11].

The Heisenberg group $\mathbf{H}^n$ is CR equivalent to the sphere $S^{2n+1} \subset \mathbf{C}^{n+1}$ minus a point via the Cayley transform, see e.g. [16], so that the Heisenberg group plays in CR geometry the same role as $\mathbf{R}^n$ in conformal geometry while the Cayley transform corresponds to the stereographic projection.

Under a change of pseudohermitian structure given by $\widetilde{\theta} = u^{p-2}\theta$ with $p = \frac{2n+2}{n}$, the pseudohermitian scalar curvature R changes according to the following equation

$$b_n \Delta_b u + Ru = \widetilde{R} u^{p-1}, \qquad b_n := \frac{2n+2}{n}, \tag{1}$$

as one can find in [16]. Here Δ_b is the sublaplacian operator on the CR manifold M, which is a linear second order subelliptic operator, see also [8], [16] and references therein. On the sublaplacian on the Heisenberg group, which we will denote by Δ_H, see also section 2.

One interesting feature of equation (1) is that the exponent p in the nonlinearity is the same as the one appearing in a Sobolev–type inequality for functions in $\mathcal{C}_0^\infty(\mathbf{H}^n)$, which is related to the CR structure defined on $\mathbf{H}^n$, that was proved by Folland and Stein in [9].

We report here a very nice table that summarizes many important similarities between CR geometry and conformal geometry, as it appears in [16].

Conformal geometry	CR geometry
Riemannian Manifold (M, g)	CR manifold (M, θ)
Euclidean space $\mathbf{R}^m$	Heisenberg group $\mathbf{H}^n$
m–sphere $S^m \subset \mathbf{R}^{m+1}$	$(2n+1)$–sphere $S^{2n+1} \subset \mathbf{C}^{n+1}$
Stereographic projection	Cayley transform
Riemannian normal coordinates	Folland-Stein normal coordinates
Scalar curvature K	Webster (pseudohermitian) scalar curvature R
Laplace-Beltrami operator Δ	Sublaplacian operator Δ_b
Sobolev spaces $W^{k,r}$	Folland-Stein spaces $S^{k,r}$
Sobolev embedding $W^{1,2} \subset L^q$, $\frac{1}{q} = \frac{1}{2} - \frac{1}{m}$	Sobolev embedding $S^{1,2} \subset L^p$, $\frac{1}{p} = \frac{1}{2} - \frac{1}{2n+2}$
Conformal change $\widetilde{g} = u^{q-2} g$	Change of contact form $\widetilde{\theta} = u^{p-2}\theta$

Conformal geometry	CR geometry
Conformal invariant $\mu(M)$	CR invariant $\lambda(M)$
Yamabe equation:	CR Yamabe equation:
$a_m \Delta u + Ku = \mu u^{q-1}$	$b_n \Delta_b u + Ru = \lambda u^{p-1}$

Many authors have already expanded the previous list with important contributions, as Gover and Graham did in [12], where they derived the CR analogues on CR manifolds of the GJMS operators defined on Riemannian manifolds. For the original result on Riemannian manifolds, see the paper by Graham, Jenne, Mason and Sparling [13].

The point of this paper is to provide a new characterization of fully nonlinear CR invariant equations of the second order on the Heisenberg group, thus adding another interesting similarity between CR geometry and conformal geometry. We then also provide comparison principles for solutions of families of fully nonlinear second order operators on $\mathbf{H}^n$, which have suitable invariances.

The original result on the Euclidean space $\mathbf{R}^n$ was proved by A. Li and the first author in [21]. There, among many other results, they showed that any fully nonlinear conformally invariant equation on $\mathbf{R}^n$ takes the form

$$F\big(x, u, \nabla u, \nabla^2 u\big) = F\Big(0, 1, 0, -\frac{n-2}{2} A^u\Big),$$

where

$$A^u := -\frac{2}{n-2} u^{-\frac{n+2}{n-2}} \nabla^2 u + \frac{2n}{(n-2)^2} u^{-\frac{2n}{n-2}} \nabla u \otimes \nabla u - \frac{2}{(n-2)^2} u^{-\frac{2n}{n-2}} |\nabla u|^2 I_n, \tag{2}$$

and $F(0,1,0,\cdot)$ is invariant under orthogonal conjugation, i.e.

$$F\Big(0, 1, 0, -\frac{n-2}{2} O^{-1} A O\Big) = F\Big(0, 1, 0, -\frac{n-2}{2} A\Big)$$

for every real symmetric $n \times n$ matrix A and real every orthogonal $n \times n$ matrix O.

The tensor A^u is very closely related to the Schouten tensor of a Riemannian manifold (M, g), which is defined by setting

$$A_g = \frac{1}{n-2}\Big(\mathrm{Ric}_g - \frac{R_g}{2(n-1)} g\Big), \tag{3}$$

where Ric_g and R_g denote the Ricci tensor and the scalar curvature associated with g, respectively. Indeed, let $g_1 = u^{\frac{4}{n-2}} g$ be a conformal change of metrics on M; then, as one can see in [28],

$$A_{g_1} = -\frac{2}{n-2} u^{-1} \nabla_g^2 u + \frac{2n}{(n-2)^2} u^{-2} \nabla_g u \otimes \nabla_g u - \frac{2}{(n-2)^2} u^{-2} |\nabla_g u|_g^2 g + A_g.$$

If one lets $g = u^{\frac{4}{n-2}} g_{\text{flat}}$, where g_{flat} denotes the Euclidean metric on $\mathbf{R}^n$, then by the above transformation formula

$$A_g = u^{\frac{4}{n-2}} A^u_{ij} dx_i dx_j,$$

where A^u is given by (2).

Lee derived in [20] the analogue transformation law for the CR Schouten tensor under a CR conformal change of the contact form θ on a CR manifold.

Remark 1.1. Let $\mathbf{N}$ denote the set of positive integers. For any $N \in \mathbf{N}$ we will denote by I_N and 0_N the identity $N \times N$ matrix and the zero $N \times N$ matrix respectively. We will denote by $\mathrm{Mat}(N, \mathbf{R})$ the set of $N \times N$ real matrices and by $\mathcal{S}^{N\times N}$ the set of real symmetric $N \times N$ matrices.

If $v, w \in \mathbf{R}^N$ for some $N \in \mathbf{N}$, we will denote by $v \otimes w$ the $N \times N$ real matrix

$$v \otimes w := [v_i w_j]_{i,j=1,\dots,N}.$$

Notice that for all $v, w \in \mathbf{R}^N$ and all $A, B \in \mathrm{Mat}(N, \mathbf{R})$ one has

$$A\big(v \otimes w\big)B = \big((Av) \otimes w\big)B = \big(Av\big) \otimes \big(B^T w\big).$$

With some abuse of notation then we will simply write it as $Av \otimes wB$.

1.1. **Fully nonlinear CR invariant equations of the second order on $\mathbf{H}^n$.** The Heisenberg group $\mathbf{H}^n$ is the set $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ endowed with the group action $\circ$ defined by

$$\xi \circ \hat{\xi} := \Big(x + \hat{x},\, y + \hat{y},\, t + \hat{t} + 2\sum_{i=1}^{n} x_i \hat{y}_i - y_i \hat{x}_i\Big) \tag{4}$$

for any $\xi = (x, y, t)$, $\hat{\xi} = (\hat{x}, \hat{y}, \hat{t})$ in $\mathbf{H}^n$, with $x = (x_1, \dots, x_n)$, $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n)$, $y = (y_1, \dots, y_n)$ and $\hat{y} = (\hat{y}_1, \dots, \hat{y}_n)$ denoting elements of $\mathbf{R}^n$. We will also use the notation $\xi = (z, t)$ with $z = x + iy$, $z \in \mathbf{C}^n \simeq \mathbf{R}^n \times \mathbf{R}^n$. Let $Q := 2n + 2$ denote the homogenous dimension of $\mathbf{H}^n$, see also [10]. We consider the norm on $\mathbf{H}^n$ defined by

$$|\xi|_{H} := \Big[\Big(\sum_{i=1}^{n} x_i^2 + y_i^2\Big)^2 + t^2\Big]^{\frac{1}{4}} = \Big(|z|^4 + t^2\Big)^{\frac{1}{4}}. \tag{5}$$

The corresponding distance on $\mathbf{H}^n$ is defined accordingly by setting

$$d_H(\xi, \hat{\xi}) := |\hat{\xi}^{-1} \circ \xi|_H$$

where $\hat{\xi}^{-1}$ is the inverse of $\hat{\xi}$ with respect to $\circ$, i.e. $\hat{\xi}^{-1} = -\hat{\xi}$. For every $\xi \in \mathbf{H}^n$ and $R > 0$ we will use the notation

$$D_R(\xi) \;:=\; \{\eta \in \mathbf{H}^n \,|\, d_H(\xi, \eta) < R\}. \tag{6}$$

For any fixed $\hat{\xi} \in \mathbf{H}^n$ we will denote by $\tau_{\hat{\xi}} : \mathbf{H}^n \to \mathbf{H}^n$ the **left translation** on $\mathbf{H}^n$ by $\hat{\xi}$, defined by

$$\tau_{\hat{\xi}}(\xi) = \hat{\xi} \circ \xi, \tag{7}$$

where $\circ$ denotes the group action defined in (4), while for any $\lambda > 0$ we will denote by $\delta_\lambda : \mathbf{H}^n \to \mathbf{H}^n$ the **dilation** defined by

$$\delta_\lambda(\xi) := (\lambda x,\, \lambda y,\, \lambda^2 t), \tag{8}$$

which satisfies

$$\delta_\lambda(\hat{\xi} \circ \xi) = \delta_\lambda(\hat{\xi}) \circ \delta_\lambda(\xi)$$

for every ξ, $\hat{\xi} \in \mathbf{H}^n$ and every $\lambda > 0$.

Notice that the norm on $\mathbf{H}^n$ defined by (5) is homogeneous of degree 1 with respect to the dilations δ_λ, i.e.

$$|\delta_\lambda(\xi)|_H = \lambda |\xi|_H \qquad \forall\, \xi \in \mathbf{H}^n,\ \lambda > 0.$$

Another group of automorphisms of $\mathbf{H}^n$ is given by the action of the n–dimensional unitary group $\mathcal{U}(n)$. Using the complex numbers notation, its action is given by

$$\rho_M(\xi) = \rho_M(z,t) := (Mz, t) \tag{9}$$

for any $M \in \mathcal{U}(n)$ and every $\xi = (z,t) \in \mathbf{H}^n$.

It is a known fact that a complex matrix $M \in \mathrm{Mat}(n, \mathbf{C})$ belongs to $\mathcal{U}(n)$, i.e. it satisfies $M \cdot (\overline{M})^T = I_n$, if and only if the block matrix $\widetilde{M} \in \mathrm{Mat}(2n, \mathbf{R})$ defined as in Theorem 1.3 by

$$\widetilde{M} := \begin{pmatrix} B & -C \\ C & B \end{pmatrix}, \qquad \text{with } B := \mathrm{Re}M,\ C := \mathrm{Im}M \in \mathrm{Mat}(n, \mathbf{R}),$$

belongs to $\mathcal{O}(2n)$, i.e. one has $\widetilde{M} \cdot (\widetilde{M})^T = I_{2n}$. Using the real numbers notation, one has

$$\rho_M(\xi) = \rho_M(x,y,t) = (Bx - Cy,\, By + Cx,\, t)$$

for any $M \in \mathcal{U}(n)$ and every $\xi = (x,y,t) \in \mathbf{H}^n$. In the case of the Euclidean space $\mathbf{R}^n$, the analogues of these maps are the usual **rotations**, given by the action of the group $\mathcal{O}(n)$ on $\mathbf{R}^n$.

We finally introduce the **inversion** map $\iota : \mathbf{H}^n \to \mathbf{H}^n$ defined by

$$\iota(\xi) = \iota(x,y,t) := (x, -y, -t) \tag{10}$$

for every $\xi = (x,y,t) \in \mathbf{H}^n$, and the map $\varphi : \mathbf{H}^n \to \mathbf{H}^n$ defined by Jerison and Lee in [16] which we shall refer to as the **CR inversion** and which is defined by the following relations:

$$\varphi(\xi) := \widetilde{\xi} \tag{11}$$

where $\widetilde{\xi} = (\widetilde{x}, \widetilde{y}, \widetilde{t})$ and

$$\widetilde{x} := \frac{xt + y|z|^2}{|\xi|_H^4}, \qquad \widetilde{y} := \frac{yt - x|z|^2}{|\xi|_H^4}, \qquad \widetilde{t} := \frac{-t}{|\xi|_H^4}. \tag{12}$$

We explicitly remark that $\left|\varphi(\xi)\right|_H = \frac{1}{|\xi|_H}$. The CR inversion of $\mathbf{H}^n$ plays the role of the usual inversion with respect to the unitary sphere in $\mathbf{R}^n$.

The elements of the group of automorphisms of $\mathbf{H}^n$ generated by the left translations (7), by the dilations (8), by the rotations (9), by the inversion map (10) and by the CR inversion (12) are called **CR maps** on $\mathbf{H}^n$.

For further references on these maps, on their definitions and their properties we also refer to the works of Koranyi [19], Jerison and Lee [16] and Birindelli and Prajapat [4], [3].

The vector fields

$$\begin{array}{rcll} X_j & := & \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, & j = 1, \ldots, n \\ Y_j & := & \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, & j = 1, \ldots, n \\ T & := & \frac{\partial}{\partial t} & \end{array} \tag{13}$$

form a base of the Lie algebra of vector fields on the Heisenberg group which are left invariant with respect to the group action $\circ$. The Heisenberg gradient, or horizontal

gradient, of a regular function $u : \mathbf{H}^n \to \mathbf{H}^n$ is then defined by

$$\nabla_H u := (X_1 u, \dots, X_n u, Y_1 u, \dots, Y_n u),$$

while its Heisenberg hessian matrix is

$$\nabla_H^2 u := \left(\begin{array}{ccc|ccc} X_1X_1u & \cdots & X_nX_1u & Y_1X_1u & \cdots & Y_nX_1u \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ X_1X_nu & \cdots & X_nX_nu & Y_1X_nu & \cdots & Y_nX_nu \\ \hline X_1Y_1u & \cdots & X_nY_1u & Y_1Y_1u & \cdots & Y_nY_1u \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ X_1Y_nu & \cdots & X_nY_nu & Y_1Y_nu & \cdots & Y_nY_nu \end{array}\right) = \left(\begin{array}{c|c} X_jX_iu & Y_jX_iu \\ \hline X_jY_iu & Y_jY_iu \end{array}\right)_{i,j=1,\dots,n} \tag{14}$$

We also define

$$\nabla_{H,s}^2 u(\xi) := \frac{1}{2}\left[\nabla_H^2 u(\xi) + \left(\nabla_H^2 u(\xi)\right)^T\right] \tag{15}$$

which is the symmetric part of the matrix $\nabla_H^2 u(\xi)$.

Now let $n \in \mathbf{N}$ and define $G, J \in \mathrm{Mat}(2n, \mathbf{R})$ by setting

$$G := \begin{pmatrix} I_n & 0_n \\ 0_n & -I_n \end{pmatrix}, \qquad J := \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}. \tag{16}$$

We notice that $\nabla_H^2 u \in \mathcal{S}^{2n\times 2n} \oplus J\mathbf{R}$, see also section 2.1.

We now introduce the definition of CR invariance for an operator F depending on ξ, u, $\nabla_H u$, $\nabla_H^2 u$. We refer to section 2 for some further basic facts concerning the Heisenberg group.

Definition 1.2. *Let $F \in \mathcal{C}^0\big(\mathbf{H}^n \times \mathbf{R}^+ \times \mathbf{R}^{2n} \times (\mathcal{S}^{2n\times 2n} \oplus J\mathbf{R})\big)$. Then $F(\cdot, u, \nabla_H u, \nabla_H^2 u)$ is CR invariant on the Heisenberg group $\mathbf{H}^n$ if for any positive function $u \in \mathcal{C}^2(\mathbf{H}^n)$ and any CR map $\psi : \mathbf{H}^n \to \mathbf{H}^n$ one has that*

$$F\big(\xi, u_\psi(\xi), \nabla_H u_\psi(\xi), \nabla_H^2 u_\psi(\xi)\big) = F\big(\psi(\xi), u(\psi(\xi)), \nabla_H u(\psi(\xi)), \nabla_H^2 u(\psi(\xi))\big), \tag{17}$$

for every ξ, where the function u_ψ is the transformed function of u through the CR map ψ, which is defined by

$$u_\psi(\xi) := |J_\psi(\xi)|^{\frac{Q-2}{2Q}} u\big(\psi(\xi)\big), \qquad \xi \in \mathbf{H}^n. \tag{18}$$

The main results of the present paper are now contained in the following theorems.

Theorem 1.3. *Let $F \in \mathcal{C}^0\big(\mathbf{H}^n \times \mathbf{R}^+ \times \mathbf{R}^{2n} \times (\mathcal{S}^{2n\times 2n} \oplus J\mathbf{R})\big)$. Then $F(\cdot, u, \nabla_H u, \nabla_H^2 u)$ is CR invariant on the Heisenberg group $\mathbf{H}^n$ if and only if*

$$F\big(\cdot, u, \nabla_H u, \nabla_H^2 u\big) \equiv F\Big(0, 1, 0, -\frac{Q-2}{2} A^u\Big), \tag{19}$$

where

$$\begin{aligned} A^u := & -\frac{2}{Q-2} u^{-\frac{Q+2}{Q-2}} \nabla_{H,s}^2 u + \frac{2Q}{(Q-2)^2} u^{-\frac{2Q}{Q-2}} \nabla_H u \otimes \nabla_H u \\ & -\frac{4}{(Q-2)^2} u^{-\frac{2Q}{Q-2}} J\nabla_H u \otimes J\nabla_H u - \frac{2}{(Q-2)^2} u^{-\frac{2Q}{Q-2}} |\nabla_H u|^2 I_{2n}, \end{aligned} \tag{20}$$

and moreover for every $A \in \mathcal{S}^{2n\times 2n} \oplus J\mathbf{R}$ one has

i) $F(0,1,0,A) = F(0,1,0,\widetilde{M}^T A\widetilde{M})$ *for every unitary matrix* $M = B + iC \in \mathcal{U}(n)$, *where we have set* $\widetilde{M} := \begin{pmatrix} B & -C \\ C & B \end{pmatrix}$,

ii) $F(0,1,0,A) = F(0,1,0,GAG)$,

iii) $F(0,1,0,A) = F(0,1,0,A+\alpha J)$ *for every* $\alpha \in \mathbf{R}$,

with J, G being defined as in (16).

We want to stress here that $A^u \in \mathcal{S}^{2n\times 2n}$, thus it always has real eigenvalues, even if the Heisenberg hessian matrix $\nabla_H^2 u$ in general is not symmetric. Let

$$\lambda(A^u) = (\lambda_1(A^u), \ldots, \lambda_{2n}(A^u))$$

denote the eigenvalues of A^u. Using Theorem 1.3 we can then provide some examples of fully nonlinear CR invariant operators of the second order on the Heisenberg group. Indeed, for $k = 1, \ldots, 2n$, let

$$\omega_k(\lambda) = \sum_{1\le i_1<\ldots<i_k\le 2n} \lambda_{i_1}\cdots\lambda_{i_k}, \qquad \lambda = (\lambda_1, \ldots, \lambda_{2n}) \in \mathbf{R}^{2n}$$

denote the k^{th} symmetric function on $\mathbf{R}^{2n}$. Then $\omega_k\big(\lambda(A^u)\big)$ is a fully nonlinear CR invariant differential operator of the second order on $\mathbf{H}^n$.

Similar operators, involving the tensors in equations (2) and (3), were studied in the context of conformal geometry on $\mathbf{R}^n$ and on more general Riemannian manifolds by many authors, see the works by Viaclovsky [28] and [29], the papers by Chang, Gursky, and Yang [6] and [7] and the works by A. Li and the first author [21] and [22], and the references therein.

1.2. **Comparison principle on a domain $\Omega \subset \mathbf{H}^n$ for fully nonlinear CR invariant equations.**

Remark 1.4. If $A, B \in \mathrm{Mat}(N, \mathbf{R})$ for some $N \in \mathbf{N}$, we will write $A \ge B$ if

$$\langle \xi, A\xi\rangle_{\mathbf{R}^N} \ge \langle \xi, B\xi\rangle_{\mathbf{R}^N} \quad \text{for every } \xi \in \mathbf{R}^N,$$

where we denoted with $\langle \cdot, \cdot\rangle_{\mathbf{R}^N}$ the usual scalar product in $\mathbf{R}^N$. If $A - B$ is a diagonalizable matrix, this is equivalent to $A - B$ having nonnegative eigenvalues, i.e. to $A - B$ being nonnegative definite.

Let $\Sigma \subset \mathcal{S}^{2n\times 2n}$ be an open set of matrices such that

$$\begin{aligned} &\text{i)} \quad A \in \overline{\Sigma},\ c \in \mathbf{R}^+ \Longrightarrow cA \in \overline{\Sigma}, \\ &\text{ii)} \quad A \in \overline{\Sigma},\ B \in \mathcal{S}^{2n\times 2n} \text{ and } B > 0 \Longrightarrow A + B \in \Sigma. \end{aligned} \tag{21}$$

Notice that condition ii) in particular implies

$$\text{iii)} \quad A \in \overline{\Sigma},\ B \in \mathcal{S}^{2n\times 2n} \text{ and } B \ge 0 \Longrightarrow A + B \in \overline{\Sigma}.$$

Theorem 1.5. *Let $\Omega \subset \mathbf{H}^n$ be a domain, let $\Sigma \subset \mathcal{S}^{2n\times 2n}$ be an open set of matrices satisfying condition* (21) *and let $u, w \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$. Assume that $u, w > 0$ on $\overline{\Omega}$, $A^u \in \overline{\Sigma}$ and $A^w \in \Sigma^c$ for every $\xi \in \Omega$. Then*

i) *if $u \ge w$ on $\partial\Omega$, $u \ge w$ in $\overline{\Omega}$,*

ii) *if* $u > w$ *on* $\partial\Omega$, $u > w$ *in* $\overline{\Omega}$.

1.3. Comparison principle for a family of fully nonlinear equations on a punctured ball $\mathbf{D_R}(\xi_0) \subset \mathbf{H^n}$.

Remark 1.6. In this section we will consider an operator $T \in \mathcal{C}^1\big(\mathbf{R}^+ \times \mathbf{R}^{2n} \times (\mathcal{S}^{2n\times 2n} \oplus J\mathbf{R})\big)$ satisfying the following assumptions

i) $T = T(s, v, U)$ is elliptic with respect to the family of vector fields $X_1, \dots, X_n$, $Y_1, \dots, Y_n$, i.e.

$$\left[-\frac{\partial T}{\partial U_{ij}}(s, v, U)\right]_{i,j=1,\dots,2n} > 0 \qquad \text{on } \mathbf{R}^+ \times \mathbf{R}^{2n} \times (\mathcal{S}^{2n\times 2n} \oplus J\mathbf{R}), \tag{22}$$

that is for every $[a, b] \subset \mathbf{R}^+$, $K_1 \subset \mathbf{R}^{2n}$ compact, $K_2 \subset \mathcal{S}^{2n\times 2n} \oplus J\mathbf{R}$ compact there exists $\beta > 0$ such that

$$\left\langle W, -\frac{\partial T}{\partial U_{ij}}(s, v, U) \cdot W \right\rangle_{\mathbf{R}^{2n}} \geq \beta |W|^2$$

for every $W \in \mathbf{R}^{2n}$, $s \in [a, b]$, $v \in K_1$, $U \in K_2$;

ii) T is invariant with respect to dilations in $\mathbf{H}^n$, i.e. for every $\lambda > 0$ and every positive function $u \in \mathcal{C}^2(\mathbf{H}^n)$ one has

$$T\big(u_{\delta_\lambda}(\xi), \nabla_H u_{\delta_\lambda}(\xi), \nabla_H^2 u_{\delta_\lambda}(\xi)\big) = T\big(u\big(\delta_\lambda(\xi)\big), \nabla_H u\big(\delta_\lambda(\xi)\big), \nabla_H^2 u\big(\delta_\lambda(\xi)\big)\big)$$

for every $\xi \in \mathbf{H}^n$.

Notice that since the operator T does not explicitly depend on $\xi \in \mathbf{H}^n$, it is automatically invariant with respect to left translations in $\mathbf{H}^n$, i.e. for every $\hat{\xi} \in \mathbf{H}^n$ and every positive function $u \in \mathcal{C}^2(\mathbf{H}^n)$ one has

$$T\big(u_{\tau_{\hat{\xi}}}(\xi), \nabla_H u_{\tau_{\hat{\xi}}}(\xi), \nabla_H^2 u_{\tau_{\hat{\xi}}}(\xi)\big) = T\big(u\big(\tau_{\hat{\xi}}(\xi)\big), \nabla_H u\big(\tau_{\hat{\xi}}(\xi)\big), \nabla_H^2 u\big(\tau_{\hat{\xi}}(\xi)\big)\big)$$

for every $\xi \in \mathbf{H}^n$.

Theorem 1.7. *Assume that the operator T satisfies the hypotheses above. Consider $D_2(\xi_0) \subset \mathbf{H}^n$ and let $u \in \mathcal{C}^2(D_2(\xi_0) \setminus \{\xi_0\})$, $w \in \mathcal{C}^2(D_2(\xi_0))$ be such that*

i) *$u > w$ in $D_2(\xi_0) \setminus \{\xi_0\}$ and $w > 0$ in $D_2(\xi_0)$,*

ii) *$\Delta_H u \leq 0$ in $D_2(\xi_0) \setminus \{\xi_0\}$,*

iii) *$T(u, \nabla_H u, \nabla_H^2 u) \geq 0 \geq T(w, \nabla_H w, \nabla_H^2 w)$ in $D_2(\xi_0) \setminus \{\xi_0\}$.*

Then $\liminf_{\xi \to \xi_0} \big(u(\xi) - w(\xi)\big) > 0$.

This result is an analogue in the Heisenberg group setting of the original result proved in the Euclidean setting by the first author (see Theorem 1.7 in [23]). From Theorem 1.3 and Theorem 1.7, the following Corollary immediately follows.

Corollary 1.8. *Let $F(A^u)$ be a CR invariant operator on the Heisenberg group, with A^u being defined as in equation* (20) *and $F \in \mathcal{C}^1(\mathcal{S}^{2n\times 2n})$. Assume that for every $A \in \mathcal{S}^{2n\times 2n}$ one has*

$$\left[\frac{\partial F}{\partial A_{ij}}(A)\right]_{i,j=1,\dots,2n} > 0.$$

Consider $D_2(\xi_0) \subset \mathbf{H}^n$ and let $u \in \mathcal{C}^2(D_2(\xi_0) \setminus \{\xi_0\})$, $v \in \mathcal{C}^2(D_2(\xi_0))$ be such that

i) $u > w$ *in* $D_2(\xi_0) \setminus \{\xi_0\}$ *and* $w > 0$ *in* $D_2(\xi_0)$,
ii) $\Delta_H u \leq 0$ *in* $D_2(\xi_0) \setminus \{\xi_0\}$,
iii) $F(A^u) \geq 0 \geq F(A^w)$ *in* $D_2(\xi_0) \setminus \{\xi_0\}$.

Then $\liminf_{\xi \to \xi_0} \big(u(\xi) - w(\xi)\big) > 0$.

The paper is organized as follows: in section 2 we introduce notations, definitions and some known facts about the Heisenberg group and the sublaplacian operator defined on it, which we are going to use throughout the paper. Section 2 also contains the formulae for $\nabla_H u_\psi$ and $\nabla^2_H u_\psi$, when $u \in \mathcal{C}^2(\mathbf{H}^n)$ and ψ is any of the generators of the group of CR maps on $\mathbf{H}^n$.

In section 3 we give the proof of Theorem 1.3, in section 4 we prove Theorem 1.5 while in section 5 we give the proof of Theorem 1.7. Section 4 also contains another result of interest, where we study "the first variation" of the operator A^u defined in (20), and which is the equivalent on the Heisenberg group of an analogous lemma proved in the Euclidean setting by the first author (see lemma 3.7 in [24]).

Finally in section 6 we collect some technical results, which are used in the previous sections.

2. Notation and preliminary facts

For future use we notice here that if we denote by $J_\psi(\xi)$ the Jacobian matrix of a CR map $\psi : \mathbf{H}^n \to \mathbf{H}^n$ evaluated at $\xi \in \mathbf{H}^n$ and by $|J_\psi(\xi)|$ its determinant, then we have

$$|J_{\tau_{\hat\xi}}(\xi)| = 1, \qquad |J_{\rho_M}(\xi)| = 1, \qquad |J_\iota(\xi)| = 1$$
$$|J_{\delta_\lambda}(\xi)| = \lambda^Q, \qquad |J_\varphi(\xi)| = \tfrac{1}{|\xi|_H^{2Q}},$$

where we recall that $Q = 2n + 2$ denote the homogenous dimension of $\mathbf{H}^n$.

Next we recall that, for any CR map ψ on $\mathbf{H}^n$ and any function $u : \mathbf{H}^n \to \mathbf{H}^n$, the transformed function u_ψ is defined as in (18) by

$$u_\psi(\xi) := |J_\psi(\xi)|^{\frac{Q-2}{2Q}} u\big(\psi(\xi)\big), \qquad \xi \in \mathbf{H}^n.$$

Then we have

$$u_{\tau_{\hat\xi}}(\xi) = u(\hat\xi \circ \xi), \quad u_{\rho_M}(\xi) = u(Bx - Cy, Cx + By, t), \quad u_\iota(\xi) = u(x, -y, -t) \tag{23}$$
$$u_{\delta_\lambda}(\xi) = \lambda^{\frac{Q-2}{2}} u(\lambda x, \lambda y, \lambda^2 t), \qquad u_\varphi(\xi) = \tfrac{1}{|\xi|_H^{Q-2}} u(\widetilde{x}, \widetilde{y}, \widetilde{t}).$$

2.1. The sublaplacian on the Heisenberg group.

Consider the vector fields X_j, Y_j for $j = 1, \dots, n$ defined in (13). The sublaplacian on the Heisenberg group is the linear differential operator of the second order defined by

$$\begin{aligned} \Delta_H u &:= \sum_{j=1}^n X_j^2 u + Y_j^2 u \\ &= \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2} + \frac{\partial^2 u}{\partial y_j^2} + 4y_j \frac{\partial^2 u}{\partial x_j \partial t} - 4x_j \frac{\partial^2 u}{\partial y_j \partial t} + 4(x_j^2 + y_j^2) \frac{\partial^2 u}{\partial t^2}. \end{aligned}$$

The sublaplacian is the trace of the Heisenberg hessian matrix defined in (14) and it is degenerate elliptic. Furthermore it has divergence form. Indeed one has

$$\Delta_H u = \mathrm{div}(A(z)\nabla u),$$

where here ∇u denotes the gradient of u in $\mathbf{R}^{2n+1}$ and

$$A(z) = A(x,y) := \begin{pmatrix} I_n & 0_n & 2y \\ 0_n & I_n & -2x \\ 2y & -2x & 4|z|^2 \end{pmatrix},$$

with I_n and 0_n denoting respectively the identity matrix and the zero matrix in $\mathbf{R}^n$.

Now notice that the smooth vector fields X_j, Y_j, $j = 1, \dots, n$ satisfy the following commutation relations

$$[X_i, Y_j] = -4T\delta_{ij}, \quad [X_i, X_j] = [Y_i, Y_j] = 0 \qquad \forall\, i, j = 1, \dots, n \tag{24}$$

and hence they and their first order commutators span the whole Lie Algebra. It is then a consequence of the classical theorem of Hörmander [14] that the sublaplacian is hypoelliptic, i.e. if $\Delta_H u \in \mathcal{C}^\infty$ then $u \in \mathcal{C}^\infty$. Moreover Δ_H satisfies the Strong Maximum Principle, as one finds in the work of Bony [5].

Remark 2.1. Notice that, by the commutation relations (24), the Heisenberg hessian matrix of a regular function u is not symmetric, in general. Indeed, it's easy to see that

$$\nabla^2_H u(\xi) \;=\; \nabla^2_{H,s} u(\xi) + 2Tu(\xi)J, \tag{25}$$

with $J \in \mathrm{Mat}(2n, \mathbf{R})$ being defined as in (16) by

$$J := \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}.$$

Remark 2.2. Using the definition of the matrix J in (16), for every regular function u one has

$$\begin{aligned}
\nabla_H u &= \nabla_z u + 2\frac{\partial u}{\partial t} Jz \\
\nabla^2_H u &= \nabla^2_z u + 2\frac{\partial}{\partial t}\nabla_z u \otimes Jz + 2Jz \otimes \frac{\partial}{\partial t}\nabla_z u + 4\frac{\partial^2 u}{\partial t^2} Jz \otimes Jz + 2\frac{\partial u}{\partial t} J \\
\nabla^2_{H,s} u &= \nabla^2_z u + 2\frac{\partial}{\partial t}\nabla_z u \otimes Jz + 2Jz \otimes \frac{\partial}{\partial t}\nabla_z u + 4\frac{\partial^2 u}{\partial t^2} Jz \otimes Jz \\
\Delta_H u &= \Delta_z u + 4|z|^2 \frac{\partial^2 u}{\partial t^2} + 4\frac{\partial}{\partial t}\langle Jz, \nabla_z u\rangle_{\mathbf{R}^{2n}},
\end{aligned}$$

where Δ_z, ∇^2_z and ∇_z are respectively the ordinary Laplace operator, the Hessian matrix and the gradient with respect to the variables $z = (x, y) \in \mathbf{R}^{2n}$. This in particular implies

$$\nabla_H u(0) = \nabla_z u(0), \qquad \nabla^2_H u(0) = \nabla^2_z u(0) + 2\frac{\partial u}{\partial t}(0) J.$$

Remark 2.3. It is not difficult to see that

$$
\begin{aligned}
\nabla_H u_{\tau_{\hat{\xi}}}(\xi) &= \nabla_H u(\hat{\xi}\circ\xi) \\
\nabla_H u_{\rho_M}(\xi) &= \widetilde{M}^T\cdot\nabla_H u(Bx-Cy, By+Cx, t) \\
\nabla_H u_\iota(\xi) &= G\cdot\nabla_H u(x,-y,-t) \\
\nabla_H u_{\delta_\lambda}(\xi) &= \lambda^{\frac{Q}{2}}\nabla_H u(\lambda x, \lambda y, \lambda^2 t),
\end{aligned} \tag{26}
$$

with $G\in \mathrm{Mat}(2n,\mathbf{R})$ being defined as in Theorem 1.3 by $G=\begin{pmatrix} I_n & 0_n \\ 0_n & -I_n\end{pmatrix}$, and that

$$
\begin{aligned}
\nabla^2_H u_{\tau_{\hat{\xi}}}(\xi) &= \nabla^2_H u(\hat{\xi}\circ\xi) \\
\nabla^2_H u_{\rho_M}(\xi) &= \widetilde{M}^T\cdot\nabla^2_H u(Bx-Cy, By+Cx, t)\cdot\widetilde{M} \\
\nabla^2_H u_\iota(\xi) &= G\cdot\nabla^2_H u(x,-y,-t)\cdot G \\
\nabla^2_H u_{\delta_\lambda}(\xi) &= \lambda^{\frac{Q+2}{2}}\nabla^2_H u(\lambda x, \lambda y, \lambda^2 t).
\end{aligned} \tag{27}
$$

Then it follows that

$$
\begin{aligned}
\Delta_H u_{\tau_{\hat{\xi}}}(\xi) &= \Delta_H u(\hat{\xi}\circ\xi) \\
\Delta_H u_{\rho_M}(\xi) &= \Delta_H u(Bx-Cy, By+Cx, t) \\
\Delta_H u_\iota(\xi) &= \Delta_H u(x,-y,-t) \\
\Delta_H u_{\delta_\lambda}(\xi) &= \lambda^{\frac{Q+2}{2}}\Delta_H u(\lambda x, \lambda y, \lambda^2 t).
\end{aligned} \tag{28}
$$

2.2. **A useful CR map.** Instead of using the CR inversion φ defined in (11) as one of the generators of the group of CR maps on $\mathbf{H}^n$, throughout the rest of the paper we will use the map $\check{\varphi} := \varphi\circ\iota$, i.e. $\check{\varphi}(\xi) = \big(\check{x},\, \check{y},\, \check{t}\big)$ for every $\xi\in\mathbf{H}^n$ with $(\check{x},\, \check{y},\, \check{t})$ being in turn defined by

$$
\check{x} = -\frac{xt+y|z|^2}{|\xi|_H^4}, \quad \check{y} = \frac{yt-x|z|^2}{|\xi|_H^4}, \quad \check{t} = \frac{t}{|\xi|_H^4}. \tag{29}
$$

We make this choice because $\check{\varphi}\big(\check{\varphi}(\xi)\big) = \xi$, while $\varphi\big(\varphi(\xi)\big) = (-x,-y,t)$ for every $\xi = (x,y,t)\in\mathbf{H}^n\setminus\{0\}$.

We now derive the transformation formulae for $\nabla_H u_{\check{\varphi}}$ and $\nabla^2_H u_{\check{\varphi}}$.

Proposition 2.4. *Let $u \in \mathcal{C}^2(\mathbf{H}^n)$, then $u_{\check{\varphi}}(\xi) = \frac{1}{|\xi|_H^{Q-2}} u(\check{x}, \check{y}, \check{t})$ for every $\xi \in \mathbf{H}^n \setminus \{0\}$. Moreover for every $j = 1, \dots, n$ and every $\xi = (x, y, t) \in \mathbf{H}^n \setminus \{0\}$ one has*

$$
\begin{aligned}
X_j u_{\check{\varphi}}(\xi) = &-(Q-2)|\xi|_H^{-(Q+2)} u\big(\check{\varphi}(\xi)\big)\big(|z|^2 x_j + t y_j\big) \\
&+|\xi|_H^{-(Q+6)} \frac{\partial u}{\partial t}\big(\check{\varphi}(\xi)\big)\big(2\big(|z|^4 - t^2\big) y_j - 4t|z|^2 x_j\big) \\
&+|\xi|_H^{-(Q+6)} \Big[\sum_{h=1}^n \frac{\partial u}{\partial x_h}\big(\check{\varphi}(\xi)\big)\Big(-\delta_{jh} t|\xi|_H^4 + 2\big(|z|^4 - t^2\big)(x_j y_h - y_j x_h) \\
&\qquad\qquad + 4t|z|^2(x_j x_h + y_j y_h)\Big)\Big] \\
&+|\xi|_H^{-(Q+6)} \Big[\sum_{h=1}^n \frac{\partial u}{\partial y_h}\big(\check{\varphi}(\xi)\big)\Big(-\delta_{jh} |z|^2|\xi|_H^4 + 2\big(|z|^4 - t^2\big)(x_j x_h + y_j y_h) \\
&\qquad\qquad + 4t|z|^2(y_j x_h - x_j y_h)\Big)\Big]
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
Y_j u_{\check{\varphi}}(\xi) = &-(Q-2)|\xi|_H^{-(Q+2)} u\big(\check{\varphi}(\xi)\big)\big(|z|^2 y_j - t x_j\big) \\
&+|\xi|_H^{-(Q+6)} \frac{\partial u}{\partial t}\big(\check{\varphi}(\xi)\big)\big(2\big(t^2 - |z|^4\big) x_j - 4t|z|^2 y_j\big) \\
&+|\xi|_H^{-(Q+6)} \Big[\sum_{h=1}^n \frac{\partial u}{\partial x_h}\big(\check{\varphi}(\xi)\big)\Big(-\delta_{jh} |z|^2|\xi|_H^4 + 2\big(|z|^4 - t^2\big)(y_j y_h + x_j x_h) \\
&\qquad\qquad + 4t|z|^2(y_j x_h - x_j y_h)\Big)\Big] \\
&+|\xi|_H^{-(Q+6)} \Big[\sum_{h=1}^n \frac{\partial u}{\partial y_h}\big(\check{\varphi}(\xi)\big)\Big(\delta_{jh} t|\xi|_H^4 + 2\big(|z|^4 - t^2\big)(y_j x_h - x_j y_h) \\
&\qquad\qquad - 4t|z|^2(x_j x_h + y_j y_h)\Big)\Big].
\end{aligned} \tag{31}
$$

Proof: By definition (18) one has $u_{\check{\varphi}}(\xi) = |J_{\check{\varphi}}(\xi)|^{\frac{Q-2}{2Q}} u(\check{\varphi}(\xi))$. Since for every $\xi \in \mathbf{H}^n$ one has that

$$
|J_{\check{\varphi}}(\xi)| = |J_{\varphi \circ \iota}(\xi)| = |J_{\varphi}(\xi)||J_{\iota}(\xi)| = |J_{\varphi}(\xi)| = \frac{1}{|\xi|_H^{2Q}},
$$

we get $u_{\check{\varphi}}(\xi) = \frac{1}{|\xi|_H^{Q-2}} u(\check{x}, \check{y}, \check{t})$ as claimed.

Then one also gets

$$
\begin{aligned}
X_j u_{\check{\varphi}}(\xi) &= u(\check{x}, \check{y}, \check{t}) X_j \left(\frac{1}{|\xi|_H^{Q-2}} \right) + \frac{1}{|\xi|_H^{Q-2}} X_j \Big(u(\check{x}, \check{y}, \check{t}) \Big) \\
&= -(Q-2)|\xi|_H^{-(Q+2)} u(\check{x}, \check{y}, \check{t})\big(|z|^2 x_j + t y_j\big) + \\
&\quad \frac{1}{|\xi|_H^{Q-2}} \Big(\sum_{h=1}^n \frac{\partial u}{\partial x_h}(\check{x}, \check{y}, \check{t}) X_j(\check{x}_h) + \sum_{h=1}^n \frac{\partial u}{\partial y_h}(\check{x}, \check{y}, \check{t}) X_j(\check{y}_h) + \frac{\partial u}{\partial t}(\check{x}, \check{y}, \check{t}) X_j(\check{t}) \Big),
\end{aligned}
$$

and in a similar way

$$\begin{aligned}
Y_j u_{\check{\varphi}}(\xi) &= u(\check{x},\check{y},\check{t})Y_j\Big(\frac{1}{|\xi|_H^{Q-2}}\Big) + \frac{1}{|\xi|_H^{Q-2}}Y_j\Big(u(\check{x},\check{y},\check{t})\Big)\\
&= -(Q-2)|\xi|_H^{-(Q+2)}u(\check{x},\check{y},\check{t})\big(|z|^2 y_j - t x_j\big) +\\
&\quad \frac{1}{|\xi|_H^{Q-2}}\Big(\sum_{h=1}^n \frac{\partial u}{\partial x_h}(\check{x},\check{y},\check{t})Y_j(\check{x}_h) + \sum_{h=1}^n \frac{\partial u}{\partial y_h}(\check{x},\check{y},\check{t})Y_j(\check{y}_h) + \frac{\partial u}{\partial t}(\check{x},\check{y},\check{t})Y_j(\check{t})\Big).
\end{aligned}$$

Now it suffices to use the formulae for $X_j(\check{x}_h)$, $X_j(\check{y}_h)$, $X_j(\check{t})$, $Y_j(\check{x}_h)$, $Y_j(\check{y}_h)$ and $Y_j(\check{t})$, which are provided in lemma 6.1 in the appendix, in order to conclude the proof. $\square$

Remark 2.5. We can rewrite formulae (30) and (31) in the following form

$$\nabla_H u_{\check{\varphi}}(\xi) = -\frac{(Q-2)}{|\xi|_H^{Q+2}}u\big(\check{\varphi}(\xi)\big)\big(|z|^2 z + tJz\big) + \frac{1}{|\xi|_H^{Q}}\,E\cdot\nabla_H u\big(\check{\varphi}(\xi)\big), \tag{32}$$

where $J \in \mathrm{Mat}(2n,\mathbf{R})$ is the matrix defined in (16) and where $E \in \mathrm{Mat}(2n,\mathbf{R})$ is the block matrix defined by

$$E := |\xi|_H^2 \left(\begin{array}{c|c} X_j(\check{x}_h) & X_j(\check{y}_h) \\ \hline Y_j(\check{x}_h) & Y_j(\check{y}_h)\end{array}\right)_{j,h=1,\dots,n}. \tag{33}$$

Using lemma 6.1 we see that the matrix E can be written in the form

$$\begin{aligned}
E &= \Bigg(-\frac{t}{|\xi|_H^2}I_{2n} + \frac{|z|^2}{|\xi|_H^2}J + \frac{1}{|\xi|_H^6}\Big(2(|z|^4 - t^2)(z\otimes Jz - Jz\otimes z) +\\
&\qquad\qquad 4t|z|^2(z\otimes z + Jz\otimes Jz)\Big)\Bigg)\cdot G\\
&= \begin{pmatrix} R & S \\ S & -R\end{pmatrix} = \begin{pmatrix} R & -S \\ S & R\end{pmatrix}\cdot G = G\cdot\begin{pmatrix} R & S \\ -S & R\end{pmatrix},
\end{aligned} \tag{34}$$

where $G \in \mathrm{Mat}(2n,\mathbf{R})$ is the matrix defined in (16) and where $R, S \in \mathrm{Mat}(n,\mathbf{R})$ are given by

$$\begin{aligned}
R &:= |\xi|_H^2\Big(X_j(\check{x}_h)\Big)_{j,h=1,\dots,n} = -|\xi|_H^2\Big(Y_j(\check{y}_h)\Big)_{j,h=1,\dots,n}\\
&= -\frac{t}{|\xi|_H^2}I_n + \frac{1}{|\xi|_H^6}\Big(2(|z|^4 - t^2)(x\otimes y - y\otimes x) + 4t|z|^2(x\otimes x + y\otimes y)\Big)\\
S &:= |\xi|_H^2\Big(X_j(\check{y}_h)\Big)_{j,h=1,\dots,n} = |\xi|_H^2\Big(Y_j(\check{x}_h)\Big)_{j,h=1,\dots,n}\\
&= -\frac{|z|^2}{|\xi|_H^2}I_n + \frac{1}{|\xi|_H^6}\Big(2(|z|^4 - t^2)(x\otimes x + y\otimes y) + 4t|z|^2(y\otimes x - x\otimes y)\Big).
\end{aligned}$$

With these definitions it's easy to see that $R + iS \in \mathcal{U}(n)$ and hence that $E \in \mathcal{O}(2n)$.

Corollary 2.6. *Let $u \in \mathcal{C}^2(\mathbf{H}^n)$, then*

$$
\begin{aligned}
\nabla^2_H u_{\check\varphi}(\xi) \quad = \quad & \frac{Q^2-4}{|\xi|_H^{Q+6}} u\big(\check\varphi(\xi)\big)\big(|z|^2 z + tJz\big) \otimes \big(|z|^2 z + tJz\big) \\
& - \frac{(Q-2)}{|\xi|_H^{Q+2}} u\big(\check\varphi(\xi)\big)\big(|z|^2 I_{2n} + tJ + 2z\otimes z + 2Jz \otimes Jz\big) \\
& - \frac{(Q-2)}{|\xi|_H^{Q+4}} \big(|z|^2 z + tJz\big) \otimes \big(E \cdot \nabla_H u\big(\check\varphi(\xi)\big)\big) \\
& - \frac{(Q-2)}{|\xi|_H^{Q+4}} \big(E \cdot \nabla_H u\big(\check\varphi(\xi)\big)\big) \otimes \big(|z|^2 z + tJz\big) \\
& + \frac{1}{|\xi|_H^{Q+2}} E \cdot \nabla^2_H u\big(\check\varphi(\xi)\big) \cdot E^T \\
& + \frac{1}{|\xi|_H^{Q-2}} \sum_{h=1}^n \nabla^2_H(\check x_h) X_h u\big(\check\varphi(\xi)\big) + \nabla^2_H(\check y_h) Y_h u\big(\check\varphi(\xi)\big)
\end{aligned} \tag{35}
$$

for every $\xi \in \mathbf{H}^n \setminus \{0\}$, where E is the matrix defined in (33).

Proof: In order to obtain formula (35) it's sufficient to recall the definition of the Heisenberg hessian matrix ∇^2_H given in formula (14) and use formula (32). $\square$

Remark 2.7. Notice that for every $\xi = (z,t) \in \mathbf{H}^n \setminus \{0\}$ one has

$$
\begin{aligned}
& \frac{1}{|\xi|_H^{Q-2}} \sum_{h=1}^n \nabla^2_H(\check x_h) X_h u\big(\check\varphi(\xi)\big) + \nabla^2_H(\check y_h) Y_h u\big(\check\varphi(\xi)\big) \\
& \quad = \frac{1}{|\xi|_H^{Q+6}} G\nabla_H u \otimes \big(2(t^2 - |z|^4)Jz + 4t|z|^2 z\big) \\
& \quad + \frac{1}{|\xi|_H^{Q+6}} \big(2(t^2 - |z|^4)Jz + 4t|z|^2 z\big) \otimes G\nabla_H u \\
& \quad + \frac{1}{|\xi|_H^{Q+6}} GJ\nabla_H u \otimes \big(2(|z|^4 - t^2)z + 4t|z|^2 Jz\big) \\
& \quad + \frac{1}{|\xi|_H^{Q+6}} \big(2(|z|^4 - t^2)z + 4t|z|^2 Jz\big) \otimes GJ\nabla_H u \\
& \quad + \frac{8}{|\xi|_H^{Q+6}} \big(\langle GJz, \nabla_H u\rangle_{\mathbf{R}^{2n}} z - \langle Gz, \nabla_H u\rangle_{\mathbf{R}^{2n}} Jz\big) \otimes (|z|^2 z - tJz) \\
& \quad + \frac{8}{|\xi|_H^{Q+6}} \big(\langle Gz, \nabla_H u\rangle_{\mathbf{R}^{2n}} z + \langle GJz, \nabla_H u\rangle_{\mathbf{R}^{2n}} Jz\big) \otimes (tz + |z|^2 Jz) \\
& \quad - \frac{16(|z|^4 - t^2)}{|\xi|_H^{Q+10}} \big(\langle GJz, \nabla_H u\rangle_{\mathbf{R}^{2n}} z - \langle Gz, \nabla_H u\rangle_{\mathbf{R}^{2n}} Jz\big) \otimes (|z|^2 z + tJz) \\
& \quad - \frac{32t|z|^2}{|\xi|_H^{Q+10}} \big(\langle Gz, \nabla_H u\rangle_{\mathbf{R}^{2n}} z + \langle GJz, \nabla_H u\rangle_{\mathbf{R}^{2n}} Jz\big) \otimes (|z|^2 z + tJz)
\end{aligned}
$$

$$+\frac{1}{|\xi|_H^{Q+6}}\big(2(|z|^4-t^2)\,\langle GJz,\,\nabla_H u\rangle_{\mathbf{R}^{2n}}+4t|z|^2\,\langle Gz,\,\nabla_H u\rangle_{\mathbf{R}^{2n}}\,\big)I_{2n}$$
$$+\frac{1}{|\xi|_H^{Q+6}}\big(2(t^2-|z|^4)\,\langle Gz,\,\nabla_H u\rangle_{\mathbf{R}^{2n}}+4t|z|^2\,\langle GJz,\,\nabla_H u\rangle_{\mathbf{R}^{2n}}\,\big)J,$$

where $G, J\in \mathrm{Mat}(2n,\mathbf{R})$ are the matrices defined in (16) and where it's understood that in the previous equality $\nabla_H u$ is to be evaluated at the point $\check{\varphi}(\xi)$.

Corollary 2.8. *Let $u\in\mathcal{C}^2(\mathbf{H}^n)$, then*

$$\Delta_H u_{\check{\varphi}}(\xi)=\frac{1}{|\xi|_H^{Q+2}}\Delta_H u(\check{x},\check{y},\check{t}). \tag{36}$$

Proof: In order to conclude one only needs to compute the trace of the matrix $\nabla_H^2 u_{\check{\varphi}}$, given in formula (35). $\square$

On the formula which is the analogue to (36) when the CR inversion φ is involved, see also [16], [3] and references therein .

Remark 2.9. By relations (28) and (36) one has

$$u_\psi^{-\frac{Q+2}{Q-2}}\Delta_H u_\psi=\big(u^{-\frac{Q+2}{Q-2}}\Delta_H u\big)\circ\psi\qquad \text{in } \mathbf{H}^n$$

for every CR map $\psi:\mathbf{H}^n\to\mathbf{H}^n$, i.e.

$$F(\cdot,u,\nabla_H u,\nabla_H^2 u):=u^{-\frac{Q+2}{Q-2}}\Delta_H u$$

satisfies definition 1.2 and it is thus a CR invariant operator on $\mathbf{H}^n$. In particular, if $u\in\mathcal{C}^2$ is a positive solution of

$$-\Delta_H u=u^{\frac{Q+2}{Q-2}}, \tag{37}$$

so is u_ψ for any CR map ψ on $\mathbf{H}^n$. Equation (37) is related to the CR Yamabe problem on $\mathbf{H}^n$, see e.g. [16].

3. Proof of Theorem 1.3

Lemma 3.1. *Given $\xi_0\in\mathbf{H}^n$, $s\in\mathbf{R}^+$, $V\in\mathbf{R}^{2n}$, $c\in\mathbf{R}$ and $S\in\mathcal{S}^{2n\times 2n}$ a symmetric $2n\times 2n$ real matrix, there exists $u\in\mathcal{C}^\infty(\mathbf{H}^n)$ which is positive and such that*

$$u(\xi_0)=s,\qquad \nabla_H u(\xi_0)=V,\qquad \nabla_H^2 u(\xi_0)=S+cJ.$$

Proof: Let $w\in\mathcal{C}^\infty(\mathbf{R}^{2n+1})$ be a positive function such that

$$w(0)=s,\qquad \nabla w(0)=\Big(V,\frac{c}{2}\Big),\qquad \nabla^2 w(0)=\left(\begin{array}{c|c} S & 0\\ \hline 0 & 1\end{array}\right).$$

Now define $u(\xi)=w(\xi_0^{-1}\circ\xi)$. Then by relations (26), (27) and remark 2.2 one has

$$\begin{aligned} u(\xi_0)&=w(0)=s\\ \nabla_H u(\xi_0)&=\nabla_H w(0)=\nabla_z w(0)=V\\ \nabla_H^2 u(\xi_0)&=\nabla_H^2 w(0)=\nabla_z^2 w(0)+2\tfrac{\partial w}{\partial t}(0)J=S+cJ\end{aligned}$$

as desired. $\square$

Proof of Theorem 1.3. We start by showing that a second order differential operator of the form (19) defined on $\mathbf{H}^n$ which satisfies the invariance properties (i), (ii) and (iii)

of the statement is CR invariant on $\mathbf{H}^n$. Indeed, using formulae (26) and (27), for every positive function $u \in C^2(\mathbf{H}^n)$ it's easy to see that on $\mathbf{H}^n$ one has

$$\begin{aligned} A^{u_{\tau_{\hat\xi}}}(\xi) &= A^u\big(\tau_{\hat\xi}(\xi)\big) && \text{for every } \hat\xi \in \mathbf{H}^n, \\ A^{u_{\rho_M}}(\xi) &= \widetilde{M}^T \cdot A^u\big(\rho_M(\xi)\big) \cdot \widetilde{M} && \text{for every } M \in \mathcal{U}(n), \\ A^{u_{\delta_\lambda}}(\xi) &= A^u\big(\delta_\lambda(\xi)\big) && \text{for every } \lambda > 0, \\ A^{u_\iota}(\xi) &= G \cdot A^u\big(\iota(\xi)\big) \cdot G. \end{aligned}$$

Moreover, using formulae (32) and (35) one can see that for every $\xi \in \mathbf{H}^n \setminus \{0\}$

$$A^{u_{\check\varphi}}(\xi) \ = \ E \cdot A^u\big(\check\varphi(\xi)\big) \cdot E^T,$$

where G, $E \in \mathrm{Mat}(2n, \mathbf{R})$ are the matrices defined respectively in (16) and in (33).

The first part of the proof is thus complete. We are now going to prove that a CR invariant differential operator of the second order on $\mathbf{H}^n$ necessarily satisfies (19) and the invariance properties (i), (ii) and (iii) of the statement.

Let $\xi_0 \in \mathbf{H}^n$, $s \in \mathbf{R}^+$, $v \in \mathbf{R}^{2n}$ and $U \in \mathcal{S}^{2n\times 2n} \oplus J\mathbf{R}$. Now consider a positive function $\phi \in \mathcal{C}^\infty(\mathbf{H}^n)$ such that $\phi(\xi_0) = s$, $\nabla_H \phi(\xi_0) = v$ and $\nabla^2_H \phi(\xi_0) = U$, see lemma 3.1. Now use relation (17) with the CR transformation

$$\psi(\xi) = \tau_{\xi_0}(\xi) = \xi_0 \circ \xi,$$

see also definition (7), and the function ϕ. By relations (23), (26) and (27) one has

$$F\big(\xi, \phi(\xi_0\circ\xi), \nabla_H\phi(\xi_0\circ\xi), \nabla^2_H\phi(\xi_0\circ\xi)\big) = F\big(\xi_0\circ\xi, \phi(\xi_0\circ\xi), \nabla_H\phi(\xi_0\circ\xi), \nabla^2_H\phi(\xi_0\circ\xi)\big).$$

Evaluating this equality in $\xi = 0$ yields

$$\begin{aligned} F(0,s,v,U) &= F\big(0, \phi(\xi_0), \nabla_H\phi(\xi_0), \nabla^2_H\phi(\xi_0)\big) \\ &= F\big(\xi_0, \phi(\xi_0), \nabla_H\phi(\xi_0), \nabla^2_H\phi(\xi_0)\big) = F(\xi_0, s, v, U). \end{aligned} \tag{38}$$

Since this holds for every $\xi_0 \in \mathbf{H}^n$, $F(\xi_0, s, v, U)$ does not depend explicitly on ξ_0. From now on we will write $F(s, v, U)$ in place of $F(\xi_0, s, v, U)$.

Now let $\phi \in \mathcal{C}^\infty(\mathbf{H}^n)$ be a positive function such that $\phi(0) = s$, $\nabla_H\phi(0) = 0$ and $\nabla^2_H\phi(0) = U$ and consider the CR transformation

$$\psi(\xi) = \rho_M(\xi) = (Bx - Cy, By + Cx, t)$$

for a matrix $M = B + iC \in \mathcal{U}(n)$, see also definition (9). Using relation (17), by formulae (23), (26) and (27) we get

$$\begin{aligned} F\big(\phi(\rho_M(\xi)),\ \widetilde{M}^T\cdot\nabla_H\phi(\rho_M(\xi)),\ \widetilde{M}^T\cdot\nabla^2_H\phi(\rho_M(\xi))\cdot\widetilde{M}\big) \\ = F\big(\phi(\rho_M(\xi)), \nabla_H\phi(\rho_M(\xi)), \nabla^2_H\phi(\rho_M(\xi))\big). \end{aligned}$$

Evaluating such equality in $\xi = 0$ yields

$$\begin{aligned} F\big(s, 0, \widetilde{M}^T U \widetilde{M}\big) &= F\big(\phi(0),\ \widetilde{M}^T\cdot\nabla_H\phi(0),\ \widetilde{M}^T\cdot\nabla^2_H\phi(0)\cdot\widetilde{M}\big) \\ &= F\big(\phi(0), \nabla_H\phi(0), \nabla^2_H\phi(0)\big) \\ &= F\big(s, 0, U\big). \end{aligned} \tag{39}$$

Consider again a positive function $\phi \in \mathcal{C}^\infty(\mathbf{H}^n)$ such that $\phi(0) = s$, $\nabla_H\phi(0) = 0$ and $\nabla^2_H\phi(0) = U$ and the CR transformation

$$\iota(\xi) = \iota(x, y, t) = (x, -y, -t),$$

see definition (10). Using relations (17), (23), (26) and (27) as in the previous cases and evaluating the resulting equality in $0 \in \mathbf{H}^n$ gives

$$F\big(s, 0, GUG\big) = F\big(s, 0, U\big). \tag{40}$$

Let $\phi \in \mathcal{C}^\infty(\mathbf{H}^n)$ be a positive function such that $\phi(0) = s$, $\nabla_H \phi(0) = 0$ and $\nabla^2_H \phi(0) = U$, let $Q = 2n + 2$ and

$$\psi(\xi) = \delta_{s^{-\frac{2}{Q-2}}}(\xi) = \Big(s^{-\frac{2}{Q-2}} x,\ s^{-\frac{2}{Q-2}} y,\ s^{-\frac{4}{Q-2}} t\Big),$$

see definition (8). By (23), (26) and (27), relation (17) yields

$$F\left(s^{-1}\phi\Big(\delta_{s^{-\frac{2}{Q-2}}}(\xi)\Big),\ s^{-\frac{Q}{Q-2}}\nabla_H\phi\Big(\delta_{s^{-\frac{2}{Q-2}}}(\xi)\Big),\ s^{-\frac{Q+2}{Q-2}}\nabla^2_H\phi\Big(\delta_{s^{-\frac{2}{Q-2}}}(\xi)\Big)\right)$$
$$= F\left(\phi\Big(\delta_{s^{-\frac{2}{Q-2}}}(\xi)\Big), \nabla_H\phi\Big(\delta_{s^{-\frac{2}{Q-2}}}(\xi)\Big), \nabla^2_H\phi\Big(\delta_{s^{-\frac{2}{Q-2}}}(\xi)\Big)\right).$$

Evaluating this equality in $\xi = 0$ gives

$$F\Big(1, 0, s^{-\frac{Q+2}{Q-2}} U\Big) = F\big(s, 0, U\big). \tag{41}$$

Now let $v = (p, q) \in \mathbf{R}^n \times \mathbf{R}^n$, $v \neq 0$ and let $\xi_0 := (x_0, y_0, t_0)$ with

$$x_0 = -\frac{(Q-2)s}{|v|^2} p, \quad y_0 = -\frac{(Q-2)s}{|v|^2} q, \quad t_0 = 0$$

and define

$$\lambda := |\xi_0|_H = \big(|x_0|^2 + |y_0|^2\big)^{\frac{1}{2}} = \frac{(Q-2)s}{|v|}.$$

Then consider a positive function $\phi \in \mathcal{C}^\infty(\mathbf{H}^n)$ such that $\phi(\xi_0) = s$, $\nabla_H\phi(\xi_0) = v$ and $\nabla^2_H\phi(\xi_0) = U$ and the CR transformation

$$\psi(\xi) = \varphi \circ \iota \circ \delta_{\lambda^{-2}}(\xi) = \big(\lambda^2 \check{x},\ \lambda^2 \check{y},\ \lambda^4 \check{t}\big), \tag{42}$$

with

$$\check{x} = -\frac{xt + y|z|^2}{|\xi|^4_H}, \quad \check{y} = \frac{yt - x|z|^2}{|\xi|^4_H}, \quad \check{t} = \frac{t}{|\xi|^4_H}, \tag{43}$$

see definitions (8), (10) and (12). Then one has $\phi_\psi(\xi) = \frac{\lambda^{Q-2}}{|\xi|_H^{Q-2}}\phi\big(\lambda^2\check{x},\ \lambda^2\check{y},\ \lambda^4\check{t}\big)$, and by relation (17) evaluated in $\psi^{-1}(\xi_0)$ one has

$$\begin{aligned} F\big(\phi_\psi(\psi^{-1}(\xi_0)), \nabla_H\phi_\psi(\psi^{-1}(\xi_0)), \nabla^2_H\phi_\psi(\psi^{-1}(\xi_0))\big) &= F\big(\phi(\xi_0), \nabla_H\phi(\xi_0), \nabla^2_H\phi(\xi_0)\big) \\ &= F(s, v, U). \end{aligned}$$

Now by Lemma 6.4 in the appendix we have that $\phi_\psi(\psi^{-1}(\xi_0)) = s$, $\nabla_H \phi_\psi(\psi^{-1}(\xi_0)) = 0$ and that

$$
\begin{aligned}
\nabla^2_H \phi_\psi(\psi^{-1}(\xi_0)) = G\Big[&- \frac{Q}{Q-2} s^{-1} Jv \otimes Jv + \frac{2}{Q-2} s^{-1} v \otimes v + \frac{1}{Q-2} s^{-1} |v|^2 I_{2n} \\
&+ J^T U J + \frac{4}{|v|^4}\Big(\langle v,\, Uv\rangle_{\mathbf{R}^{2n}}\, Jv \otimes Jv + \langle Jv,\, UJv\rangle_{\mathbf{R}^{2n}}\, v \otimes v \\
&- \langle Jv, Uv\rangle_{\mathbf{R}^{2n}}\, v \otimes Jv - \langle v, UJv\rangle_{\mathbf{R}^{2n}} Jv \otimes v\Big) + \frac{2}{|v|^2}\Big(Jv \otimes J^T U^T v \\
&- v \otimes J^T U^T Jv + J^T U v \otimes Jv - J^T U J v \otimes v\Big)\Big] G,
\end{aligned} \tag{44}
$$

with G, J being defined as in (16) and where we recall that $\langle v_1,\, v_2\rangle_{\mathbf{R}^{2n}}$ denotes the scalar product of the vectors $v_1, v_2 \in \mathbf{R}^{2n}$. Then

$$
F\big(s, 0, \nabla^2_H \phi_\psi(\psi^{-1}(\xi_0))\big) \;=\; F(s, v, U). \tag{45}
$$

Now consider the matrix $E \in \mathcal{O}(2n)$ defined in (33). It's not difficult to see that, when evaluated at the point $\psi^{-1}(\xi_0)$,

$$
E = G\left(\frac{2}{|v|^2}\big(Jv \otimes v - v \otimes Jv\big) + J^T \right).
$$

Notice that we can apply results (39) and (40) to relation (45) using the orthogonal matrix E, since it can be written in the form $G\widetilde{M}$ with

$$
M = B + iC \;\in \mathcal{U}(n), \qquad \widetilde{M} = \begin{pmatrix} B & -C \\ C & B \end{pmatrix}
$$

and

$$
B = \frac{2}{|v|^2}(q \otimes p - p \otimes q), \qquad C = -\frac{2}{|v|^2}(p \otimes p + q \otimes q) + I_n.
$$

We conclude that

$$
F\big(s, v, U\big) = F\big(s, 0, \nabla^2_H \phi_\psi(\psi^{-1}(\xi_0))\big) = F\big(s, 0, E^T\, \nabla^2_H \phi_\psi(\psi^{-1}(\xi_0))\, E\big). \tag{46}
$$

An easy calculation then yields

$$
E^T\, \nabla^2_H \phi_\psi(\psi^{-1}(\xi_0))\, E = -\frac{Q}{Q-2} s^{-1} v \otimes v + \frac{2}{Q-2} s^{-1} Jv \otimes Jv + \frac{1}{Q-2} s^{-1} |v|^2 I_{2n} + U
$$

and hence from relation (46) we get

$$
F\big(s, v, U\big) = F\left(s, 0, -\tfrac{Q}{Q-2} s^{-1} v \otimes v + \tfrac{2}{Q-2} s^{-1} Jv \otimes Jv + \tfrac{1}{Q-2} s^{-1} |v|^2 I_{2n} + U\right).
$$

This equality trivially holds also in the case $v = 0$, then for every $s \in \mathbf{R}^+$, $v \in \mathbf{R}^{2n}$, $U \in \mathcal{S}^{2n\times 2n} \oplus J\mathbf{R}$ by (41) one has

$$
\begin{aligned}
&F\big(s, v, U\big) = \\
&\quad F\left(1, 0, -\tfrac{Q}{Q-2} s^{-\frac{2Q}{Q-2}} v \otimes v + \tfrac{2}{Q-2} s^{-\frac{2Q}{Q-2}} Jv \otimes Jv + \tfrac{1}{Q-2} s^{-\frac{2Q}{Q-2}} |v|^2 I_{2n} + s^{-\frac{Q+2}{Q-2}} U\right).
\end{aligned} \tag{47}
$$

Now let $\lambda > 0$ and consider the CR map ψ defined in (42). Let $\xi_0 = (0, 0, \lambda^2) \in \mathbf{H}^n$ and pick a positive function $\phi \in \mathcal{C}^\infty(\mathbf{H}^n)$ such that $\phi(\xi_0) = 1$, $\nabla_H \phi(\xi_0) = 0$ and $\nabla^2_H \phi(\xi_0) = U$.

By Lemma 6.5 in the appendix we have that $\phi_\psi(\psi^{-1}(\xi_0)) = 1$, $\nabla_H \phi_\psi(\psi^{-1}(\xi_0)) = 0$ and $\nabla_H^2 \phi_\psi(\psi^{-1}(\xi_0)) = -\frac{Q-2}{\lambda^2} J + GUG$. Since F is CR invariant on $\mathbf{H}^n$, we have

$$\begin{aligned}
F(1,0,U) &= F\big(\phi(\xi_0), \nabla_H \phi(\xi_0), \nabla_H^2 \phi(\xi_0)\big) \\
&= F\big(\phi_\psi(\psi^{-1}(\xi_0)), \nabla_H \phi_\psi(\psi^{-1}(\xi_0)), \nabla_H^2 \phi_\psi(\psi^{-1}(\xi_0))\big) \\
&= F\Big(1, 0, -\frac{Q-2}{\lambda^2} J + GUG\Big).
\end{aligned}$$

We now use (40) together with the relation

$$G \cdot \Big[-\frac{Q-2}{\lambda^2} J + GUG \Big] \cdot G \;=\; \frac{Q-2}{\lambda^2} J + U$$

to conclude that for every $\lambda > 0$ one has

$$F(1,0,U) = F\Big(1, 0, \frac{Q-2}{\lambda^2} J + U\Big).$$

Since $\lambda > 0$ is arbitrary, we have that for every $\alpha > 0$

$$F(1,0,U) = F\Big(1, 0, \alpha J + U\Big). \tag{48}$$

If we consider $\lambda > 0$, the CR map ψ defined in (42), the point $\xi_0 = (0, 0, -\lambda^2) \in \mathbf{H}^n$ and a positive function $\phi \in \mathcal{C}^\infty(\mathbf{H}^n)$ such that $\phi(\xi_0) = 1$, $\nabla_H \phi(\xi_0) = 0$ and $\nabla_H^2 \phi(\xi_0) = U$, we can use Lemma 6.6 and repeat the above argument. Thus we obtain

$$F(1,0,U) = F\Big(1, 0, -\alpha J + U\Big) \tag{49}$$

for every $\alpha > 0$. From equations (48) and (49) then we get

$$F(1,0,U) = F\Big(1, 0, \alpha J + U\Big) \tag{50}$$

for every $\alpha \in \mathbf{R}$. Now notice that

$$U = \Big[\frac{U + U^T}{2}\Big] + \Big[\frac{U - U^T}{2}\Big] = \Big[\frac{U + U^T}{2}\Big] + \alpha J$$

for a suitable $\alpha \in \mathbf{R}$. Then by relations (47) and (50) we finally get that for every $s \in \mathbf{R}^+$, $v \in \mathbf{R}^{2n}$, $U \in \mathcal{S}^{2n \times 2n} \oplus J\mathbf{R}$ one has

$$F\big(s, v, U\big) = F\left(1, 0, -\frac{Q-2}{2} A\big(s, v, U\big)\right),$$

with

$$\begin{aligned}
A\big(s, v, U\big) \;:=\; & \frac{2Q}{(Q-2)^2} s^{-\frac{2Q}{Q-2}} v \otimes v - \frac{4}{(Q-2)^2} s^{-\frac{2Q}{Q-2}} Jv \otimes Jv \\
& - \frac{2}{(Q-2)^2} s^{-\frac{2Q}{Q-2}} |v|^2 I_{2n} - \frac{2}{(Q-2)} s^{-\frac{Q+2}{Q-2}} \Big[\frac{U + U^T}{2}\Big].
\end{aligned}$$

Theorem 1.3 is now proved. $\square$

Remark 3.2. Notice that for every positive function $u \in \mathcal{C}^2(\mathbf{H}^n)$ one has

$$\operatorname{trace}\big(A^u\big) = -\frac{2}{Q-2} u^{-\frac{Q+2}{Q-2}} \Delta_{H} u$$

which, modulo the harmless constant $-\frac{2}{Q-2}$, is the example recalled in remark 2.9.

4. **Proof of Theorem 1.5**

We start this section with a lemma which will be needed in the course of the proof of Theorem 1.5.

For $u \in \mathcal{C}^2(\Omega)$, $u > 0$ define $\phi := u^{-\frac{2}{Q-2}}$. Then $\phi \in \mathcal{C}^2(\Omega)$, $\phi > 0$ and one has

$$A^u = A_\phi := \phi \nabla^2_{H,s} \phi - \tfrac{1}{2} \big|\nabla_{H} \phi\big|^2 I_{2n} - J\nabla_{H}\phi \otimes J\nabla_{H}\phi. \tag{51}$$

Lemma 4.1. *Let $\Omega \subset \mathbf{H}^n$ be a bounded open domain, $\phi \in \mathcal{C}^2(\Omega)$ and $\phi > 0$ in Ω. Let $\eta(\xi) = \eta(z,t) := e^{\delta|z|^2}$ for $\delta > 0$. Then there exists $\bar\delta > 0$, depending only on $\sup_\Omega |z|$, such that $\forall\, \delta \in (0, \bar\delta)$ and $\forall\, \varepsilon > 0$ one has*

$$A_{\phi+\varepsilon\eta} \geq \Big(1 + \varepsilon\frac{\eta}{\phi}\Big) A_\phi + \varepsilon\delta\eta\phi I_{2n} \qquad \text{in } \Omega. \tag{52}$$

Proof: For every $\phi, \eta \in \mathcal{C}^2(\Omega)$ with $\phi, \eta > 0$ in Ω and every $\varepsilon > 0$ one has

$$\begin{aligned} A_{\phi+\varepsilon\eta} &= (\phi+\varepsilon\eta)\nabla^2_{H,s}(\phi+\varepsilon\eta) - \frac{1}{2}\big|\nabla_{H}(\phi+\varepsilon\eta)\big|^2 I_{2n} - J\nabla_{H}(\phi+\varepsilon\eta) \otimes J\nabla_{H}(\phi+\varepsilon\eta) \\ &= A_\phi + \varepsilon\Big(\phi\nabla^2_{H,s}\eta + \eta\nabla^2_{H,s}\phi - \langle\nabla_{H}\phi,\, \nabla_{H}\eta\rangle_{\mathbf{R}^{2n}} I_{2n} \\ &\qquad\qquad - J\nabla_{H}\phi \otimes J\nabla_{H}\eta - J\nabla_{H}\eta \otimes J\nabla_{H}\phi\Big) + \varepsilon^2 A_\eta. \end{aligned}$$

By definition (51) then one gets

$$\begin{aligned} A_{\phi+\varepsilon\eta} &= \Big(1+\varepsilon\frac{\eta}{\phi}\Big)A_\phi + \varepsilon\Big(\phi\nabla^2_{H,s}\eta + \frac{\big|\nabla_{H}\phi\big|^2}{2\phi}\eta I_{2n} + \frac{\eta}{\phi} J\nabla_{H}\phi \otimes J\nabla_{H}\phi \\ &\quad - \langle\nabla_{H}\phi,\, \nabla_{H}\eta\rangle_{\mathbf{R}^{2n}} I_{2n} - J\nabla_{H}\phi \otimes J\nabla_{H}\eta - J\nabla_{H}\eta \otimes J\nabla_{H}\phi\Big) + \varepsilon^2 A_\eta. \end{aligned} \tag{53}$$

Now notice that

$$\nabla_{H}\Big(\frac{\eta}{\phi}\Big) \otimes \nabla_{H}\Big(\frac{\eta}{\phi}\Big) = \frac{1}{\phi^2}\nabla_{H}\eta \otimes \nabla_{H}\eta + \frac{\eta}{\phi^3}\Big(\frac{\eta}{\phi}\nabla_{H}\phi \otimes \nabla_{H}\phi - \nabla_{H}\phi \otimes \nabla_{H}\eta - \nabla_{H}\eta \otimes \nabla_{H}\phi\Big),$$

then it follows that

$$\begin{aligned} &\frac{\phi^3}{\eta}\left(J\nabla_{H}\Big(\frac{\eta}{\phi}\Big) \otimes J\nabla_{H}\Big(\frac{\eta}{\phi}\Big) - \frac{1}{\phi^2} J\nabla_{H}\eta \otimes J\nabla_{H}\eta\right) = \\ &\qquad \left(\frac{\eta}{\phi} J\nabla_{H}\phi \otimes J\nabla_{H}\phi - J\nabla_{H}\phi \otimes J\nabla_{H}\eta - J\nabla_{H}\eta \otimes J\nabla_{H}\phi\right). \end{aligned} \tag{54}$$

Inserting relation (54) into (53) we get

$$A_{\phi+\varepsilon\eta} = \Big(1+\varepsilon\frac{\eta}{\phi}\Big)A_\phi + \varepsilon\Big(\phi\nabla^2_{H,s}\eta + \frac{|\nabla_H\phi|^2}{2\phi}\eta I_{2n} - \langle\nabla_H\phi,\,\nabla_H\eta\rangle_{\mathbf{R}^{2n}} I_{2n} \\ \frac{\phi^3}{\eta}J\nabla_H\Big(\frac{\eta}{\phi}\Big)\otimes J\nabla_H\Big(\frac{\eta}{\phi}\Big) - \frac{\phi}{\eta}J\nabla_H\eta\otimes J\nabla_H\eta\Big) + \varepsilon^2 A_\eta. \tag{55}$$

Now let $\eta(\xi)=\eta(z,t):=e^{\delta|z|^2}$, with $\delta>0$ to be chosen later. Then for every $\xi\in\mathbf{H}^n$ one has $\eta(\xi)\geq 1$ and

$$\nabla_H\eta(\xi)=2\delta\eta(\xi)z,\qquad \nabla^2_{H,s}\eta(\xi)=2\delta\eta(\xi)I_{2n}+4\delta^2\eta(\xi)z\otimes z. \tag{56}$$

By relations (56) and since

$$0_{2n}\leq v\otimes v\leq |v|^2 I_{2n}\qquad\text{for all } v\in\mathbf{R}^{2n},$$

it follows that if $0<\delta\leq\frac{1}{8}\big(\sup_\Omega|z|\big)^{-2}$ one has

$$\begin{aligned} A_\eta &= 2\delta\eta^2 I_{2n}+4\delta^2\eta^2 z\otimes z-2\delta^2\eta^2|z|^2 I_{2n}-4\delta^2\eta^2 Jz\otimes Jz\\ &\geq 2\delta\eta^2 I_{2n}-2\delta^2\eta^2|z|^2 I_{2n}-4\delta^2\eta^2|z|^2 I_{2n}\\ &\geq \frac{5}{4}\delta\eta^2 I_{2n} \end{aligned} \tag{57}$$

and hence A_η is nonnegative definite in Ω. Moreover one also has

$$\begin{aligned} &\phi\nabla^2_{H,s}\eta+\frac{|\nabla_H\phi|^2}{2\phi}\eta I_{2n}-\langle\nabla_H\phi,\nabla_H\eta\rangle_{\mathbf{R}^{2n}} I_{2n}+\tfrac{\phi^3}{\eta}J\nabla_H\Big(\tfrac{\eta}{\phi}\Big)\otimes J\nabla_H\Big(\tfrac{\eta}{\phi}\Big)-\tfrac{\phi}{\eta}J\nabla_H\eta\otimes J\nabla_H\eta\\ &\quad\geq \phi\nabla^2_{H,s}\eta+\frac{|\nabla_H\phi|^2}{2\phi}\eta I_{2n}-\langle\nabla_H\phi,\,\nabla_H\eta\rangle_{\mathbf{R}^{2n}} I_{2n}-\tfrac{\phi}{\eta}J\nabla_H\eta\otimes J\nabla_H\eta\\ &\quad= 2\delta\eta\phi I_{2n}+4\delta^2\eta\phi\, z\otimes z+\frac{|\nabla_H\phi|^2}{2\phi}\eta I_{2n}-2\delta\eta\,\langle\nabla_H\phi,\,z\rangle_{\mathbf{R}^{2n}} I_{2n}-4\delta^2\eta\phi Jz\otimes Jz\\ &\quad\geq 2\delta\eta\phi I_{2n}+\frac{|\nabla_H\phi|^2}{2\phi}\eta I_{2n}-\frac{|\nabla_H\phi|^2}{4\phi}\eta I_{2n}-4\delta^2\eta\phi|z|^2 I_{2n}-4\delta^2\eta\phi|z|^2 I_{2n}\\ &\quad\geq\Big(\delta\eta\phi+\frac{|\nabla_H\phi|^2}{4\phi}\eta\Big)I_{2n} \end{aligned}$$

and hence

$$\begin{aligned} &\phi\nabla^2_{H,s}\eta+\frac{|\nabla_H\phi|^2}{2\phi}\eta I_{2n}-\langle\nabla_H\phi,\nabla_H\eta\rangle_{\mathbf{R}^{2n}} I_{2n}\\ &\qquad+\tfrac{\phi^3}{\eta}J\nabla_H\Big(\tfrac{\eta}{\phi}\Big)\otimes J\nabla_H\Big(\tfrac{\eta}{\phi}\Big)-\tfrac{\phi}{\eta}J\nabla_H\eta\otimes J\nabla_H\eta\geq\delta\eta\phi I_{2n}. \end{aligned} \tag{58}$$

Then from (55) by relations (57) and (58) we have

$$A_{\phi+\varepsilon\eta}\geq\Big(1+\varepsilon\frac{\eta}{\phi}\Big)A_\phi+\varepsilon\delta\eta\phi I_{2n}\qquad\text{in }\Omega.$$

The proof of the lemma is now complete. □

Remark 4.2. Inequality (52) is equivalent to

$$A^{\left[\left(u^{-\frac{2}{Q-2}}+\varepsilon\eta\right)^{-\frac{Q-2}{2}}\right]}\ \geq\ \Big(1+\varepsilon\eta u^{\frac{2}{Q-2}}\Big)A^u+\varepsilon\delta\eta u^{-\frac{2}{Q-2}}I_{2n}\qquad\text{in }\Omega.$$

We are now ready to start with the proof of Theorem 1.5.

Proof of Theorem 1.5: We start by proving part (i) of the statement.

Let $\phi := u^{-\frac{2}{Q-2}}$ and $\theta := w^{-\frac{2}{Q-2}}$. Then $u(\xi) \geq w(\xi)$ if and only if $\phi(\xi) \leq \theta(\xi)$, for any $\xi \in \overline{\Omega}$. Hence in particular we have $\phi \leq \theta$ on $\partial\Omega$. Moreover we also have

$$A_\phi = A^u \in \overline{\Sigma}, \qquad A_\theta = A^w \in \Sigma^c \quad \text{in } \Omega. \tag{59}$$

Now by contradiction suppose there exists $\xi_0 \in \Omega$ such that $u(\xi_0) < w(\xi_0)$, i.e. such that $\phi(\xi_0) > \theta(\xi_0)$. Multiply ϕ by a constant $\alpha_* \in (0,1)$ so that

$$\begin{array}{rcll} \theta & > & \alpha_*\phi & \text{on } \partial\Omega, \\ \theta & \geq & \alpha_*\phi & \text{in } \Omega, \\ \theta(\xi_1) & = & \alpha_*\phi(\xi_1) & \text{for some } \xi_1 \in \Omega. \end{array} \tag{60}$$

One can easily prove that $\alpha_* = \sup\big\{\alpha \in (0,1) : \beta\phi \leq \theta \text{ in } \Omega, \forall\, \beta \in (0,\alpha)\big\}$. Now use lemma 4.1, and let $\eta(\xi) = e^{\delta|z|^2}$ for some $\delta > 0$ small enough so that for $\varepsilon > 0$ one has

$$A_{\alpha_*\phi+\varepsilon\eta} \geq \Big(1 + \varepsilon\frac{\eta}{\alpha_*\phi}\Big)A_{\alpha_*\phi} + \varepsilon\delta\eta(\alpha_*\phi)I_{2n} \qquad \text{in } \Omega. \tag{61}$$

Choose $\varepsilon > 0$ small enough, so that

$$\theta > \alpha_*\phi + \varepsilon\eta \qquad \text{on } \partial\Omega.$$

For instance this can be achieved by choosing $0 < \varepsilon < \frac{1}{2}\Big(\inf_{\partial\Omega}(\theta - \alpha_*\phi)\Big)\Big(\sup_{\partial\Omega}\eta\Big)^{-1}$. Then we have $\theta(\xi_1) < \alpha_*\phi(\xi_1) + \varepsilon\eta(\xi_1)$.

Now, in a similar way as we already did with α_* in relations (60), let $\gamma \in (0,1)$ be such that

$$\begin{array}{rcll} \theta & > & \gamma\big(\alpha_*\phi + \varepsilon\eta\big) & \text{on } \partial\Omega, \\ \theta & \geq & \gamma\big(\alpha_*\phi + \varepsilon\eta\big) & \text{in } \Omega, \\ \theta(\xi_2) & = & \gamma\big(\alpha_*\phi(\xi_2) + \varepsilon\eta(\xi_2)\big) & \text{for some } \xi_2 \in \Omega, \end{array} \tag{62}$$

where $\gamma = \sup\big\{c \in (0,1) : \beta\big(\alpha_*\phi + \varepsilon\eta\big) \leq \theta \text{ in } \Omega, \forall\, \beta \in (0,c)\big\}$. Consider the CR map $\tau_{\xi_2}(\xi)$ and the transformed functions $\phi_{\tau_{\xi_2}}$, $\theta_{\tau_{\xi_2}}$ and $\eta_{\tau_{\xi_2}}$, see also definition (7) and relation (23). By relations (62) then we have

$$\begin{array}{rcll} \theta_{\tau_{\xi_2}} & \geq & \gamma\big(\alpha_*\phi_{\tau_{\xi_2}} + \varepsilon\eta_{\tau_{\xi_2}}\big) & \text{in } \tau_{\xi_2}^{-1}(\overline{\Omega}), \\ \theta_{\tau_{\xi_2}}(0) & = & \gamma\big(\alpha_*\phi_{\tau_{\xi_2}}(0) + \varepsilon\eta_{\tau_{\xi_2}}(0)\big). & \end{array} \tag{63}$$

Then we have

$$\nabla\theta_{\tau_{\xi_2}}(0) \;=\; \gamma\big(\alpha_*\nabla\phi_{\tau_{\xi_2}}(0) + \varepsilon\nabla\eta_{\tau_{\xi_2}}(0)\big), \tag{64}$$

$$\nabla^2\theta_{\tau_{\xi_2}}(0) \;\geq\; \gamma\big(\alpha_*\nabla^2\phi_{\tau_{\xi_2}}(0) + \varepsilon\nabla^2\eta_{\tau_{\xi_2}}(0)\big), \tag{65}$$

where we recall that ∇ and ∇^2 denote respectively the gradient and the Hessian matrix of a regular function in $\mathbf{R}^{2n+1}$. Now recall that by remark 2.2 for any function $f \in \mathcal{C}^1(\Omega)$ one has $\nabla_H f(0) = \nabla_z f(0)$, hence by relation (64) we have

$$\nabla_H\theta_{\tau_{\xi_2}}(0) = \gamma\big(\alpha_*\nabla_H\phi_{\tau_{\xi_2}}(0) + \varepsilon\nabla_H\eta_{\tau_{\xi_2}}(0)\big). \tag{66}$$

We notice also that (65) implies

$$\nabla^2_{H,s}\theta_{\tau_{\xi_2}}(0) \geq \gamma\big(\alpha_*\nabla^2_{H,s}\phi_{\tau_{\xi_2}}(0) + \varepsilon\nabla^2_{H,s}\eta_{\tau_{\xi_2}}(0)\big). \tag{67}$$

Indeed for any $z \in \mathbf{R}^{2n}$ let $\zeta := (z,0) \in \mathbf{R}^{2n+1}$, then we have

$$\Big\langle z,\, \nabla_z^2 \theta_{\tau_{\xi_2}}(0)\, z \Big\rangle_{\mathbf{R}^{2n}} = \sum_{i,j=1}^{2n} \big[\nabla_z^2 \theta_{\tau_{\xi_2}}(0)\big]_{ij} z_i z_j$$

$$= \sum_{i,j=1}^{2n+1} \big[\nabla^2 \theta_{\tau_{\xi_2}}(0)\big]_{ij} \zeta_i \zeta_j = \Big\langle \zeta,\, \nabla^2 \theta_{\tau_{\xi_2}}(0)\, \zeta \Big\rangle_{\mathbf{R}^{2n+1}}$$

$$\Big\langle z,\, \gamma\big(\alpha_* \nabla_z^2 \phi_{\tau_{\xi_2}}(0) + \varepsilon \nabla_z^2 \eta_{\tau_{\xi_2}}(0)\big)\, z \Big\rangle_{\mathbf{R}^{2n}} = \Big\langle \zeta,\, \gamma\big(\alpha_* \nabla^2 \phi_{\tau_{\xi_2}}(0) + \varepsilon \nabla^2 \eta_{\tau_{\xi_2}}(0)\big)\, \zeta \Big\rangle_{\mathbf{R}^{2n+1}}$$

Now, by relation (65), this in turn implies that

$$\nabla_z^2 \theta_{\tau_{\xi_2}}(0) \geq \gamma\big(\alpha_* \nabla_z^2 \phi_{\tau_{\xi_2}}(0) + \varepsilon \nabla_z^2 \eta_{\tau_{\xi_2}}(0)\big). \tag{68}$$

By remark 2.2 one has $\nabla_{H,s}^2 f(0) = \nabla_z^2 f(0)$ for every function $f \in \mathcal{C}^2(\Omega)$, so that inequality (68) finally implies (67).

But then by formulae (26) and (27) we have

$$\begin{aligned}
\theta(\xi_2) = \theta_{\tau_{\xi_2}}(0) &= \gamma\big(\alpha_* \phi_{\tau_{\xi_2}}(0) + \varepsilon \eta_{\tau_{\xi_2}}(0)\big) = \gamma\big(\alpha_* \phi(\xi_2) + \varepsilon \eta(\xi_2)\big) && \text{by (63)},\\
\nabla_H \theta(\xi_2) = \nabla_H \theta_{\tau_{\xi_2}}(0) &= \gamma\big(\alpha_* \nabla_H \phi_{\tau_{\xi_2}}(0) + \varepsilon \nabla_H \eta_{\tau_{\xi_2}}(0)\big) && \text{by (66)}\\
&= \gamma\big(\alpha_* \nabla_H \phi(\xi_2) + \varepsilon \nabla_H \eta(\xi_2)\big),\\
\nabla_{H,s}^2 \theta(\xi_2) = \nabla_{H,s}^2 \theta_{\tau_{\xi_2}}(0) &\geq \gamma\big(\alpha_* \nabla_{H,s}^2 \phi_{\tau_{\xi_2}}(0) + \varepsilon \nabla_{H,s}^2 \eta_{\tau_{\xi_2}}(0)\big) && \text{by (67)}\\
&= \gamma\big(\alpha_* \nabla_{H,s}^2 \phi(\xi_2) + \varepsilon \nabla_{H,s}^2 \eta(\xi_2)\big).
\end{aligned}$$

This in turn implies that at $\xi_2 \in \Omega$ one has

$$\begin{aligned}
A_\theta &= \theta \nabla_{H,s}^2 \theta - \frac{1}{2}\big|\nabla_H \theta\big|^2 I_{2n} - J\nabla_H \theta \otimes J\nabla_H \theta\\
&\geq A_{\gamma(\alpha_*\phi + \varepsilon\eta)}\\
&= \gamma^2 A_{\alpha_*\phi+\varepsilon\eta}\\
&\geq \gamma^2 \Big(\Big(1 + \varepsilon \frac{\eta}{\alpha_*\phi}\Big) A_{\alpha_*\phi} + \varepsilon\delta\eta(\alpha_*\phi) I_{2n}\Big) \qquad \text{by (61)}\\
&= \gamma^2 \alpha_*^2 \Big(1 + \varepsilon \frac{\eta}{\alpha_*\phi}\Big) A_\phi + \varepsilon\delta\gamma^2\alpha_*\eta\phi I_{2n}.
\end{aligned} \tag{69}$$

Since $A_\phi \in \overline{\Sigma}$ and since $\gamma^2\alpha_*^2\Big(1 + \varepsilon\frac{\eta}{\alpha_*\phi}\Big) > 0$ in Ω, by condition (21) we get

$$\gamma^2 \alpha_*^2 \Big(1 + \varepsilon \frac{\eta}{\alpha_*\phi}\Big) A_\phi \in \overline{\Sigma}. \tag{70}$$

Then in $\xi_2 \in \Omega$ we have

$$A_\theta = \Big(A_\theta - \gamma^2\alpha_*^2\Big(1 + \varepsilon\frac{\eta}{\alpha_*\phi}\Big)A_\phi\Big) + \gamma^2\alpha_*^2\Big(1 + \varepsilon\frac{\eta}{\alpha_*\phi}\Big)A_\phi := B + cA_\phi,$$

with $cA_\phi \in \overline{\Sigma}$ by (70) and with $B \in \mathcal{S}^{2n\times 2n}$, $B \geq \varepsilon\delta\gamma^2\alpha_*\eta\phi I_{2n} > 0$ by (69).

By condition (21) it follows that $A_\theta \in \Sigma$ when evaluated in $\xi_2 \in \Omega$, which contradicts our hypothesis $A_\theta \in \Sigma^c$ in Ω, see condition (59). Then we have $u \geq w$ in $\overline{\Omega}$.

Part (i) of the statement of the theorem is thus proved. Now we turn our attention to part (ii).

Consider again $\phi := u^{-\frac{2}{Q-2}}$ and $\theta := w^{-\frac{2}{Q-2}}$. Then $A_\phi \in \overline{\Sigma}$ and $A_\theta \in \Sigma^c$. Since $u > w$ on $\partial\Omega$, we have $\theta > \phi$ on $\partial\Omega$. By part (i) we have $u \geq w$ in $\overline{\Omega}$, and hence $\theta \geq \phi$ in $\overline{\Omega}$.

Now suppose by contradiction that there exists $\xi_1 \in \Omega$ such that $u(\xi_1) = w(\xi_1)$, i.e. $\theta(\xi_1) = \phi(\xi_1)$. Then we have condition (60), this time with $\alpha_* = 1$.

The proof now proceeds as in the previous case (i), where one has just to substitute $\alpha_* = 1$ in all the calculations. Then we can conclude that at some point $\xi_2 \in \Omega$ one has $A_\theta \in \Sigma$, which again contradicts our hypothesis $A_\theta \in \Sigma^c$ in Ω. Thus $u > w$ in $\overline{\Omega}$, and the proof of the theorem is now complete. $\square$

Remark 4.3. Notice that, by choosing

$$\Sigma := \left\{ A \in \mathcal{S}^{2n\times 2n} \,\middle|\, \mathrm{tr} A > 0 \right\}$$

in Theorem 1.5, we have the following corollary:

Let $u, w \in \mathcal{C}^2(\Omega) \cup \mathcal{C}^0(\overline{\Omega})$ with $\Omega \subset \mathbf{H}^n$ a bounded open domain. Suppose that $u, w > 0$ in $\overline{\Omega}$, and that $\Delta_H u \leq 0$, $\Delta_H w \geq 0$ in Ω. Then

i) if $u \geq w$ on $\partial\Omega$, $u \geq w$ in $\overline{\Omega}$,
ii) if $u > w$ on $\partial\Omega$, $u > w$ in $\overline{\Omega}$.

This is also a consequence of the weak maximum principle applied to the sublaplacian on the Heisenberg group.

5. **Proof of Theorem 1.7**

We start this section with some results that will be needed in the course of the proof of Theorem 1.7.

Theorem 5.1. *Let $n \geq 1$, $Q = 2n + 2$ and consider $D_1(0) \subset \mathbf{H}^n$, $u \in \mathcal{C}^2\big(D_1(0) \setminus \{0\}\big)$ such that*

$$\Delta_H u \leq 0 \qquad \text{in } D_1(0) \setminus \{0\}.$$

Let $f, g : D_1(0) \to \mathbf{R}$ be bounded functions such that

i) $f(0) = g(0)$,
ii) *f, g are differentiable in 0 and $\nabla_H f(0) \neq \nabla_H g(0)$,*
iii) *$u(\xi) \geq f(\xi)$, $u(\xi) \geq g(\xi)$ for every $\xi \in D_1(0) \setminus \{0\}$.*

Then one has $\displaystyle\lim_{R\to 0^+} \inf_{D_R(0)} u > f(0)$.

Proof: For every $\xi \in D_1(0) \setminus \{0\}$ define

$$\widetilde{u}(\xi) \;:=\; u(\xi) - f(0) - \langle \nabla f(0), \xi \rangle_{\mathbf{R}^{2n+1}} \,,$$

and for $\xi \in D_1(0)$ define

$$\begin{aligned} \widetilde{f}(\xi) &:= f(\xi) - f(0) - \langle \nabla f(0), \xi \rangle_{\mathbf{R}^{2n+1}} \,, \\ \widetilde{g}(\xi) &:= g(\xi) - f(0) - \langle \nabla f(0), \xi \rangle_{\mathbf{R}^{2n+1}} \,, \end{aligned}$$

where ∇ denotes the usual gradient of a function defined in a domain of $\mathbf{R}^{2n+1}$. Then $\widetilde{u}$, $\widetilde{f}$, $\widetilde{g}$ satisfy all of the hypotheses of the theorem, moreover with

$$\widetilde{f}(0) = \widetilde{g}(0) = 0, \quad \nabla \widetilde{f}(0) = 0, \quad \nabla_H \widetilde{f}(0) = 0, \quad \nabla_H \widetilde{g}(0) \neq 0.$$

If we prove that the result of the theorem holds for the functions $\widetilde{u}$, $\widetilde{f}$, $\widetilde{g}$, that is if we have

$$\lim_{R\to 0}\inf_{D_R(0)} \widetilde{u} > 0 = \widetilde{f}(0),$$

then we also have

$$\lim_{R\to 0}\Big(\inf_{D_R(0)} u - f(0) - \inf_{D_R(0)} \langle \nabla f(0),\, \xi\rangle_{\mathbf{R}^{2n+1}}\Big) \geq \lim_{R\to 0}\inf_{D_R(0)} \widetilde{u} > 0.$$

Thus we obtain the desired result on u, i.e. $\lim_{R\to 0}\inf_{D_R(0)} u > f(0)$.

Hence without loss of generality we can assume that the functions u, f, g also satisfy

$$f(0) = g(0) = 0, \quad \nabla f(0) = 0, \quad \nabla_H f(0) = 0, \quad \nabla_H g(0) \neq 0, \tag{71}$$

and thus we have to prove that $\lim_{R\to 0}\inf_{D_R(0)} u > 0 = f(0)$.

Conditions (71) in particular imply that as $|\xi| \to 0$ one has

$$f(\xi) = o(|\xi|), \qquad g(\xi) = \langle \nabla g(0),\, \xi\rangle_{\mathbf{R}^{2n+1}} + o(|\xi|),$$

where $|\cdot|$ denotes the usual Euclidean norm. Then in $D_1(0)$ we have

$$u(\xi) \geq -h(\xi), \qquad u(\xi) \geq \langle \nabla g(0),\, \xi\rangle_{\mathbf{R}^{2n+1}} - h(\xi) := \langle \zeta,\, \xi\rangle_{\mathbf{R}^{2n+1}} - h(\xi) \tag{72}$$

with $\zeta = (z_1, t_1) := \nabla g(0) \in \mathbf{R}^{2n} \times \mathbf{R}$, $z_1 = \nabla_H g(0) \neq 0$ by (71), and with $h(\xi) \geq 0$ bounded and $h(\xi) = o(|\xi|)$ as $|\xi| \to 0$. Now define on $D_1(0) \setminus \{0\}$ the function

$$u_\lambda(\xi) := \frac{1}{\lambda} u\big(\delta_\lambda(\xi)\big),$$

with $0 < \lambda \ll 1$ and $\delta_\lambda(\xi)$ defined by relation (8). Notice that, as λ tends to 0, we have on $\overline{D_1(0)}$

$$\frac{1}{\lambda} h\big(\delta_\lambda(\xi)\big) = \frac{1}{\lambda}|\delta_\lambda(\xi)|\, o(1) = \frac{\sqrt{\lambda^2|z|^2 + \lambda^4 t^2}}{\lambda}\, o(1) \leq |\xi|\, o(1) \leq \sqrt{2}\, o(1).$$

Hence $h\big(\delta_\lambda(\xi)\big) = o(\lambda)$ as $\lambda \to 0$, uniformly for $\xi \in \overline{D_1(0)}$. By relations (72) then we have

$$u_\lambda(\xi) \geq -\frac{1}{\lambda} h\big(\delta_\lambda(\xi)\big) \geq -o(1) \qquad \text{on } \overline{D_1(0)} \setminus \{0\} \tag{73}$$

$$\begin{aligned} u_\lambda(\xi) &\geq \frac{1}{\lambda}\langle \zeta,\, \delta_\lambda(\xi)\rangle_{\mathbf{R}^{2n+1}} - \frac{1}{\lambda} h\big(\delta_\lambda(\xi)\big) \\ &\geq \langle z_1,\, z\rangle_{\mathbf{R}^{2n}} + \lambda t_1 t - o(1) \geq \langle z_1,\, z\rangle_{\mathbf{R}^{2n}} - 0(1) \qquad \text{on } \overline{D_1(0)} \setminus \{0\}, \end{aligned} \tag{74}$$

uniformly in $\xi = (z, t)$ as $\lambda \to 0$.

By (73) for every $\varepsilon > 0$ there exists $\lambda_0 > 0$ such that

$$u_\lambda(\xi) \geq -\varepsilon \qquad \text{in } \overline{D_1(0)} \setminus \{0\} \text{ for all } \lambda < \lambda_0. \tag{75}$$

Moreover, if we set $\xi_0 := \left(\frac{z_1}{2|z_1|}, 0\right)$, we have $|\xi_0|_H = |\xi_0| = \frac{1}{2}$ and it's easy to see that $\overline{D_{\frac{1}{4}}(\xi_0)} \subset D_1(0) \setminus \{0\}$. By (74) on $\overline{D_{\frac{1}{4}}(\xi_0)}$ then we have

$$
\begin{aligned}
u_\lambda(\xi) \;&\geq\; \langle z_1, z\rangle_{\mathbf{R}^{2n}} - o(1) = \frac{1}{2}|z_1| + \left\langle z_1, z - \frac{z_1}{2|z_1|}\right\rangle_{\mathbf{R}^{2n}} - o(1) \\
&\geq\; \frac{1}{2}|z_1| - |z_1|\left|z - \frac{z_1}{2|z_1|}\right| - o(1) \\
&\geq\; \frac{1}{2}|z_1| - |z_1||\xi - \xi_0|_H - o(1) \\
&\geq\; \frac{1}{2}|z_1| - \frac{1}{4}|z_1| - o(1) > c_0,
\end{aligned} \tag{76}
$$

with $c_0 := \frac{1}{8}|z_1| > 0$, for every positive λ which is smaller than a suitable $\lambda_1 > 0$. From now on we will assume that $0 < \lambda < \bar{\lambda} := \min\{\lambda_0, \lambda_1\}$.

Now let $\sigma^{(\varepsilon)}$ be the solution of

$$
\begin{cases}
\Delta_H \sigma^{(\varepsilon)} = 0 & \text{on } D_1(0) \setminus \overline{D_{\frac{1}{4}}(\xi_0)} \\
\sigma^{(\varepsilon)} = -2\varepsilon & \text{on } \partial D_1(0) \\
\sigma^{(\varepsilon)} = \frac{1}{2}c_0 & \text{on } \partial D_{\frac{1}{4}}(\xi_0)
\end{cases} \tag{77}
$$

Since $\Omega := D_1(0) \setminus \overline{D_{\frac{1}{4}}(\xi_0)}$ is a smooth domain and since its boundary is characteristic for Δ_H only in the north and south poles of the two balls, i.e. in

$$
N_1 = (0,1), \;\; S_1 = (0,-1), \;\; N_2 = \left(\frac{\zeta_z}{2|\zeta_z|}, \frac{1}{16}\right), \;\; S_2 = \left(\frac{\zeta_z}{2|\zeta_z|}, -\frac{1}{16}\right),
$$

problem (77) admits a unique solution which is $\mathcal{C}^\infty$ in the interior of the domain, because the operator is hypoelliptic since it satisfies Hörmander's condition (see [14]), and also up to the boundary in any point which is different from N_1, S_1, N_2, S_2 (see [18] and [15]). The unique solution of problem (77) is also continuous up to the boundary in all the points, see [15].

By the maximum principle, see [5], the solution depends continuously on the data of the problem and as ε tends to 0 we have

$$
\sup_{\overline{\Omega}} \left|\sigma^{(\varepsilon)} - \sigma^{(0)}\right| \to 0, \tag{78}
$$

where $\sigma^{(0)}$ is the unique solution of

$$
\begin{cases}
\Delta_H \sigma^{(0)} = 0 & \text{on } D_1(0) \setminus \overline{D_{\frac{1}{4}}(\xi_0)} \\
\sigma^{(0)} = 0 & \text{on } \partial D_1(0) \\
\sigma^{(0)} = \frac{1}{2}c_0 & \text{on } \partial D_{\frac{1}{4}}(\xi_0).
\end{cases} \tag{79}
$$

By the strong maximum principle, see [5], we have that $\sigma^{(0)} \geq 0$ in $\overline{\Omega}$ and $\sigma^{(0)}$ cannot attain its minimum value in Ω. In particular we have $\sigma^{(0)}(0) > 0$. Then by (78) if we choose $\varepsilon > 0$ small enough we have

$$
\sigma^{(\varepsilon)}(0) \geq \frac{1}{2}\sigma^{(0)}(0) > 0. \tag{80}
$$

Notice moreover that by the maximum principle one also has

$$-2\varepsilon \leq \sigma^{(\varepsilon)} \leq \frac{1}{2}c_0 \qquad \text{on } \overline{\Omega}. \tag{81}$$

Now fix $\varepsilon > 0$ such that (80) holds, and for $0 < \lambda < \bar{\lambda}$ and any $0 < r < \frac{1}{8}$ define

$$\Theta_\lambda(\xi) = u_\lambda(\xi) + Ar^{Q-2}\big(|\xi|_H^{-(Q-2)} - 1\big) - \sigma^{(\varepsilon)}(\xi) \tag{82}$$

on $D_1(0) \setminus \overline{\big(D_r(0) \cup D_{\frac{1}{4}}(\xi_0)\big)}$, with $A > 1$ to be fixed later. Notice that $\overline{D_r(0)} \subset D_1(0)$ and that $\overline{D_r(0)} \cap \overline{D_{\frac{1}{4}}(\xi_0)} = \emptyset$. Moreover one also has $\Delta_H\big(|\xi|_H^{-(Q-2)}\big) = 0$ in $\mathbf{H}^n \setminus \{0\}$, since $|\xi|_H^{-(Q-2)}$ is a constant multiple of the fundamental solution of Δ_H centered at 0.

Then $\Delta_H \Theta_\lambda = \Delta_H u_\lambda \leq 0$ in $D_1(0) \setminus \overline{\big(D_r(0) \cup D_{\frac{1}{4}}(\xi_0)\big)}$.

Now notice that $\sigma^{(\varepsilon)} = -2\varepsilon$ on $\partial D_1(0)$, and hence $\sigma^{(\varepsilon)} \leq -\frac{3}{2}\varepsilon$ near $\partial D_1(0)$. But then near $\partial D_1(0)$ by (75) one has

$$\Theta_\lambda \geq u_\lambda - \sigma^{(\varepsilon)} \geq -\varepsilon + \frac{3}{2}\varepsilon = \frac{1}{2}\varepsilon > 0.$$

On $\partial D_{\frac{1}{4}}(\xi_0)$, by (76) and (77), one has

$$\Theta_\lambda \geq u_\lambda - \sigma^{(\varepsilon)} \geq c_0 - \frac{1}{2}c_0 = \frac{1}{2}c_0 > 0.$$

Finally on $\partial D_r(0)$, by (82) and (81), one has

$$\Theta_\lambda \geq -\varepsilon + Ar^{Q-2}\big(r^{-(Q-2)} - 1\big) - \frac{1}{2}c_0 \geq A\left(1 - \frac{1}{8^{Q-2}}\right) - \varepsilon - \frac{1}{2}c_0 > 0$$

if we choose $A > 1$ large enough. Hence $\Theta_\lambda > 0$ on $\partial\Big(D_1(0) \setminus \overline{\big(D_r(0) \cup D_{\frac{1}{4}}(\xi_0)\big)}\Big)$. From the maximum principle then it follows that $\Theta_\lambda > 0$ on the whole domain.

Now fix $\xi \in D_1(0) \setminus \{0\}$ and let $0 < r < \min\big\{|\xi|_H, \frac{1}{8}\big\}$. For all $0 < \lambda < \bar{\lambda}$ then we have

$$u_\lambda(\xi) + Ar^{Q-2}\big(|\xi|_H^{-(Q-2)} - 1\big) - \sigma^{(\varepsilon)}(\xi) > 0,$$

and by letting r tend to 0 we get $u_\lambda(\xi) \geq \sigma^{(\varepsilon)}(\xi)$.

Then for all $R \in (0, \lambda)$ one has

$$\inf_{D_R(0)} u(\xi) = \inf_{D_{\frac{R}{\lambda}}(0)} u\big(\delta_\lambda(\xi)\big) = \lambda \inf_{D_{\frac{R}{\lambda}}(0)} u_\lambda(\xi) \geq \lambda \inf_{D_{\frac{R}{\lambda}}(0)} \sigma^{(\varepsilon)}(\xi).$$

Hence by the continuity of $\sigma^{(\varepsilon)}$ and by relation (80) we get

$$\lim_{R\to 0} \inf_{D_R(0)} u(\xi) \geq \lambda \lim_{R\to 0} \inf_{D_{\frac{R}{\lambda}}(0)} \sigma^{(\varepsilon)}(\xi) = \lambda\, \sigma^{(\varepsilon)}(0) \geq \frac{\lambda}{2}\sigma^{(0)}(0) > 0.$$

Thus the proof of the theorem is now complete. $\square$

Exploiting left translations and dilations, defined respectively in (7) and (8), one can easily derive the following Corollary from Theorem 5.1.

Corollary 5.2. *Let $n \geq 1$, $Q = 2n + 2$, $R > 0$, $\xi_0 \in \mathbf{H}^n$ and consider $D_R(\xi_0) \subset \mathbf{H}^n$, $u \in \mathcal{C}^2\big(D_R(\xi_0) \setminus \{\xi_0\}\big)$ such that*

$$\Delta_H u \leq 0 \qquad \text{in } D_R(\xi_0) \setminus \{\xi_0\}.$$

Let $f, g : D_R(\xi_0) \to \mathbf{R}$ be bounded functions such that
i) $f(\xi_0) = g(\xi_0)$,
ii) *f, g are differentiable in ξ_0 and $\nabla_H f(\xi_0) \neq \nabla_H g(\xi_0)$,*
iii) *$u(\xi) \geq f(\xi)$, $u(\xi) \geq g(\xi)$ for every $\xi \in D_R(\xi_0) \setminus \{\xi_0\}$.*

Then one has $\lim_{r\to 0^+} \inf_{D_r(\xi_0)} u > f(\xi_0)$.

For any positive function $w \in \mathcal{C}^2\big(D_2(0)\big)$, $\xi \in D_2(0)$ and $\lambda > 0$ let

$$w^{\xi,\lambda}(\eta) = w_{\tau_\xi \circ \delta_\lambda}(\eta) = \lambda^{\frac{Q-2}{2}} w\big(\xi \circ \delta_\lambda(\eta)\big), \tag{83}$$

for $\eta \in \mathbf{H}^n$ such that $\xi \circ \delta_\lambda(\eta) \in D_2(0)$.

Lemma 5.3. *Assume $u \in \mathcal{C}^2\big(D_2(0) \setminus \{0\}\big)$, $w \in \mathcal{C}^2\big(D_2(0)\big)$, $u > w$ in $D_2(0) \setminus \{0\}$, $w > 0$ in $D_2(0)$. Then there exists $\varepsilon_1 \in (0, 1)$ such that*

$$w^{\xi,1-\sqrt{\varepsilon}}(\eta) < u(\eta)$$

for every $\xi, \eta \in \mathbf{H}^n$ and $\varepsilon > 0$ such that $|\xi|_H < \varepsilon < \varepsilon_1$ and $0 < |\eta|_H \leq 1$.

Proof: Let γ, $\varepsilon_0 \in (0, 1)$ be small constants to be chosen later. Let ξ, $\eta \in \mathbf{H}^n$ and $\varepsilon > 0$ be such that $|\xi|_H < \varepsilon < \varepsilon_0$ and $0 < |\eta|_H \leq \gamma$. Then one has

$$|\xi \circ \delta_{1-\sqrt{\varepsilon}}(\eta)|_H \leq |\xi|_H + (1 - \sqrt{\varepsilon})|\eta|_H < \varepsilon + (1 - \sqrt{\varepsilon})\gamma < 2$$

and hence in particular $\xi \circ \delta_{1-\sqrt{\varepsilon}}(\eta) \in D_2(0)$. Moreover for a suitable constant $C > 0$ one has

$$|\xi \circ \delta_{1-\sqrt{\varepsilon}}(\eta) - \eta| \leq C\sqrt{\varepsilon}(\sqrt{\varepsilon} + \gamma). \tag{84}$$

Since $w \in \mathcal{C}^2\big(D_2(0)\big)$, by relation (84) for ε_0 and δ small enough we have that

$$w\big(\xi \circ \delta_{1-\sqrt{\varepsilon}}(\eta)\big) = w(\eta) + O\big(|\xi \circ \delta_{1-\sqrt{\varepsilon}}(\eta) - \eta|\big) = w(\eta) + \sqrt{\varepsilon} O(\sqrt{\varepsilon} + \gamma),$$

uniformly in ξ, η. Then

$$\begin{aligned} w^{\xi,1-\sqrt{\varepsilon}}(\eta) - u(\eta) &< w^{\xi,1-\sqrt{\varepsilon}}(\eta) - w(\eta) \\ &= \left(1 - \frac{Q-2}{2}\sqrt{\varepsilon} + O(\varepsilon)\right) w\big(\xi \circ \delta_{1-\sqrt{\varepsilon}}(\eta)\big) - w(\eta) \\ &= \left(1 - \frac{Q-2}{2}\sqrt{\varepsilon} + O(\varepsilon)\right) \big(w(\eta) + \sqrt{\varepsilon} O(\sqrt{\varepsilon} + \gamma)\big) - w(\eta) \\ &= \left[-\frac{Q-2}{2} w(\eta) + O(\sqrt{\varepsilon} + \gamma)\right] \sqrt{\varepsilon}, \end{aligned}$$

uniformly in ξ, η. Thus, for some ε_0, $\gamma > 0$ small enough we get

$$w^{\xi,1-\sqrt{\varepsilon}}(\eta) < u(\eta) \qquad \text{if} \quad |\xi|_H < \varepsilon < \varepsilon_0 \quad \text{and} \quad 0 < |\eta|_H \leq \gamma. \tag{85}$$

Now let ξ, $\eta \in \mathbf{H}^n$ and $\varepsilon > 0$ be such that $|\xi|_H < \varepsilon < \varepsilon_0$ and $0 < |\eta|_H \leq 1$. Then one finds

$$|\xi \circ \delta_{1-\sqrt{\varepsilon}}(\eta) - \eta| \leq C\sqrt{\varepsilon}.$$

Since $w \in \mathcal{C}^2\big(D_2(0)\big)$ one has

$$w\big(\xi \circ \delta_{1-\sqrt{\varepsilon}}(\eta)\big) = w(\eta) + O\big(|\xi \circ \delta_{1-\sqrt{\varepsilon}}(\eta) - \eta|\big) = w(\eta) + O(\sqrt{\varepsilon}),$$

uniformly in ξ, η. Choose $\varepsilon_1 \in (0, \varepsilon_0)$ such that

$$O(\sqrt{\varepsilon_1}) < \min_{\gamma \le |\zeta|_H \le 1} [u(\zeta) - w(\zeta)].$$

This is possible by our hypotheses on the functions u, w. Now for every ξ, $\eta \in \mathbf{H}^n$ such that $|\xi|_H < \varepsilon < \varepsilon_1$ and $\gamma \le |\eta|_H \le 1$ we have

$$w^{\xi, 1-\sqrt{\varepsilon}}(\eta) = w(\eta) + O(\sqrt{\varepsilon}) < w(\eta) + \big(u(\eta) - w(\eta)\big) = u(\eta). \tag{86}$$

Relations (85) and (86) together give the desired result. □

Lemma 5.4. *Assume $u \in \mathcal{C}^2\big(D_2(0) \setminus \{0\}\big)$, $w \in \mathcal{C}^2\big(D_2(0)\big)$, $u > w$ in $D_2(0) \setminus \{0\}$, $w > 0$ in $D_2(0)$. Then there exists $\varepsilon_2 \in (0,1)$ such that for every $\varepsilon \in (0, \varepsilon_2)$ one has*

$$w^{\xi,\lambda}(\eta) < u(\eta)$$

for every $\xi \in \mathbf{H}^n$ such that $|\xi|_H < \varepsilon$, every $\eta \in \mathbf{H}^n$ such that $|\eta|_H = 1$ and every $\lambda \in [1 - \sqrt{\varepsilon}, 1 + \sqrt{\varepsilon}]$.

Proof: Let

$$\varepsilon_0 := \inf_{\partial D_1(0)} [u(\zeta) - w(\zeta)],$$

then $\varepsilon_0 > 0$. For every $\varepsilon \in (0, 1/9)$, every $\lambda \in [1 - \sqrt{\varepsilon}, 1 + \sqrt{\varepsilon}]$ and every ξ, $\eta \in \mathbf{H}^n$ such that $|\xi|_H < \varepsilon$ and $|\eta|_H = 1$ one has $\xi \circ \delta_\lambda(\eta) \in D_{\frac{3}{2}}(0)$. Thus we get

$$\begin{aligned} w^{\xi,\lambda}(\eta) - u(\eta) &= \big[w\big(\xi \circ \delta_\lambda(\eta)\big) - w(\eta)\big] + \big[w(\eta) - u(\eta)\big] + \big(\lambda^{\frac{Q-2}{2}} - 1\big)\, w\big(\xi \circ \delta_\lambda(\eta)\big) \\ &\le \big|w\big(\xi \circ \delta_\lambda(\eta)\big) - w(\eta)\big| - \inf_{\partial D_1(0)} \big[u(\zeta) - w(\zeta)\big] + \big|\lambda^{\frac{Q-2}{2}} - 1\big| \sup_{\overline{D_{\frac{3}{2}}(0)}} w \\ &\le \big|w\big(\xi \circ \delta_\lambda(\eta)\big) - w(\eta)\big| - \varepsilon_0 + C_1\sqrt{\varepsilon} \end{aligned} \tag{87}$$

for some constant $C_1 > 0$.

Now notice that for every $\varepsilon \in (0,1)$, every $\lambda \in [1 - \sqrt{\varepsilon}, 1 + \sqrt{\varepsilon}]$ and every ξ, $\eta \in \mathbf{H}^n$ such that $|\xi|_H < \varepsilon$ and $|\eta|_H = 1$ one has

$$|\xi \circ \delta_\lambda(\eta) - \eta| < C_2\sqrt{\varepsilon} \tag{88}$$

for some constant $C_2 > 0$. By the uniform continuity of w on $\overline{D_{\frac{3}{2}}(0)}$, since we have η, $\big(\xi \circ \delta_\lambda(\eta)\big) \in \overline{D_{\frac{3}{2}}(0)}$, by (88) we can find $\varepsilon_* \in (0, 1/9)$ small enough so that for every $\varepsilon \in (0, \varepsilon_*)$ we have

$$\big|w\big(\xi \circ \delta_\lambda(\eta)\big) - w(\eta)\big| < \frac{\varepsilon_0}{2}.$$

Now let $\varepsilon_2 = \min\big\{\varepsilon_*, \frac{\varepsilon_0^2}{16\, C_1^2}\big\}$. By inequality (87), for every $\varepsilon \in (0, \varepsilon_2)$, every $\lambda \in [1 - \sqrt{\varepsilon}, 1 + \sqrt{\varepsilon}]$, every ξ, $\eta \in \mathbf{H}^n$ such that $|\xi|_H < \varepsilon$ and $|\eta|_H = 1$ we have

$$w^{\xi,\lambda}(\eta) - u(\eta) < -\frac{\varepsilon_0}{2} + C_1\sqrt{\varepsilon} < -\frac{\varepsilon_0}{4} < 0.$$

The proof of the lemma is now complete. □

Lemma 5.5. *Assume $u \in \mathcal{C}^2\big(D_2(0)\setminus\{0\}\big)$, $w \in \mathcal{C}^2\big(D_2(0)\big)$, $u > w$ in $D_2(0)\setminus\{0\}$, $w > 0$ in $D_2(0)$. If*

$$\liminf_{\xi\to 0}[u(\xi) - w(\xi)] = 0,$$

then there exists $\varepsilon_3 \in (0,1)$ such that

$$\sup_{0<|\eta|_H\le 1} \left\{ w^{\xi,1+\frac{\sqrt{\varepsilon}}{2}}(\eta) - u(\eta) \right\} > 0$$

for every $\xi \in \mathbf{H}^n$ and every $\varepsilon > 0$ such that $|\xi|_H < \varepsilon < \varepsilon_3$.

Proof: For $\varepsilon_3 > 0$ small enough and every $\varepsilon \in (0, \varepsilon_3)$, if $|\xi|_H < \varepsilon < \varepsilon_3$ one has

$$\begin{aligned}
\limsup_{\eta\to 0} \left\{ w^{\xi,1+\frac{\sqrt{\varepsilon}}{2}}(\eta) - u(\eta) \right\} &= w^{\xi,1+\frac{\sqrt{\varepsilon}}{2}}(0) - \liminf_{\eta\to 0} u(\eta) \\
&= w^{\xi,1+\frac{\sqrt{\varepsilon}}{2}}(0) - w(0) \\
&= \left[1 + \frac{Q-2}{2}\,\frac{\sqrt{\varepsilon}}{2} + O(\varepsilon)\right] w(\xi) - w(0).
\end{aligned} \tag{89}$$

Now notice that $|\xi| \le \sqrt{2}\varepsilon$. Since $w \in \mathcal{C}^2\big(D_2(0)\big)$, we have

$$w(\xi) = w(0) + O(|\xi|) = w(0) + O(\varepsilon).$$

Thus by relation (89) we get

$$\begin{aligned}
\limsup_{\eta\to 0} \left\{ w^{\xi,1+\frac{\sqrt{\varepsilon}}{2}}(\eta) - u(\eta) \right\} &= \left[1 + \frac{Q-2}{2}\,\frac{\sqrt{\varepsilon}}{2} + O(\varepsilon)\right] \big(w(0) + O(\varepsilon)\big) - w(0) \\
&= \left(\frac{Q-2}{4} w(0) + O(\sqrt{\varepsilon})\right)\sqrt{\varepsilon} > 0,
\end{aligned}$$

if $\varepsilon \in (0, \varepsilon_3)$ and ε_3 is small enough. Thus it follows that

$$\sup_{0<|\eta|_H\le 1} \left\{ w^{\xi,1+\frac{\sqrt{\varepsilon}}{2}}(\eta) - u(\eta) \right\} > 0. \qquad \square$$

We are now ready to prove Theorem 1.7.

Proof of Theorem 1.7: Without loss of generality, up to a translation we can assume that $\xi_0 = 0$.

Arguing by contradiction, suppose that the conclusion of the theorem does not hold, i.e. that

$$\liminf_{\xi\to 0}[u(\xi) - w(\xi)] = 0. \tag{90}$$

Then let $\varepsilon := \frac{1}{8}\min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, with ε_1, ε_2, ε_3 being provided by Lemma 5.3, Lemma 5.4 and Lemma 5.5 respectively. By Lemma 5.3 one has

$$w^{\xi,1-\sqrt{\varepsilon}}(\eta) < u(\eta) \qquad \text{for all } \eta \in \overline{D_1(0)} \setminus \{0\}, \tag{91}$$

for every $\xi \in D_\varepsilon(0)$. Now for any such ξ define

$$\overline{\lambda}(\xi) := \sup\left\{ \mu \ge 1-\sqrt{\varepsilon} \;\middle|\; w^{\xi,\lambda}(\eta) < u(\eta) \quad \forall\, \eta \in \overline{D_1(0)}\setminus\{0\}, \text{ for all } \lambda \in [1-\sqrt{\varepsilon}, \mu] \right\}.$$

By relation (91) we have that $\overline{\lambda}(\xi)$ is well defined and that $\overline{\lambda}(\xi) \geq 1-\sqrt{\varepsilon}$, for every $\xi \in D_\varepsilon(0)$. By Lemma 5.5 we also have that $\overline{\lambda}(\xi) < 1+\frac{\sqrt{\varepsilon}}{2}$ for every $\xi \in D_\varepsilon(0)$.

Now by the definition of $\overline{\lambda}(\xi)$ one has that

$$w^{\xi,\overline{\lambda}(\xi)}(\eta) \leq u(\eta) \qquad \text{for every } \eta \in \overline{D_1(0)} \setminus \{0\} \text{ and every } \xi \in D_\varepsilon(0). \tag{92}$$

Moreover, since $\overline{\lambda}(\xi) \in [1-\sqrt{\varepsilon}, 1+\frac{\sqrt{\varepsilon}}{2}]$, by Lemma 5.4 one also has that

$$w^{\xi,\overline{\lambda}(\xi)}(\eta) < u(\eta) \qquad \text{for every } \eta \in \partial D_1(0) \text{ and every } \xi \in D_\varepsilon(0). \tag{93}$$

We make the following *claim:*

$$w^{\xi,\overline{\lambda}(\xi)}(\eta) < u(\eta) \qquad \text{for every } \eta \in \overline{D_1(0)} \setminus \{0\} \text{ and every } \xi \in D_\varepsilon(0). \tag{94}$$

For every $\xi \in D_\varepsilon(0)$, $\eta \in \overline{D_1(0)} \setminus \{0\}$ we have $\xi \circ \delta_{\overline{\lambda}(\xi)}(\eta) \in D_{\frac{3}{2}}(0)$. By the invariance of the operator T, we have that

$$T\big(w^{\xi,\overline{\lambda}(\xi)}, \nabla_H w^{\xi,\overline{\lambda}(\xi)}, \nabla_H^2 w^{\xi,\overline{\lambda}(\xi)}\big)(\eta) \equiv T\big(w, \nabla_H w, \nabla_H^2 w\big)\big(\xi \circ \delta_{\overline{\lambda}(\xi)}(\eta)\big) \leq 0 \quad \text{for } \eta \in \overline{D_1(0)}$$

and thus

$$T\big(u, \nabla_H u, \nabla_H^2 u\big) - T\big(w^{\xi,\overline{\lambda}(\xi)}, \nabla_H w^{\xi,\overline{\lambda}(\xi)}, \nabla_H^2 w^{\xi,\overline{\lambda}(\xi)}\big) \geq 0 \qquad \text{on } \overline{D_1(0)} \setminus \{0\}.$$

Now let Ω be an open, connected set such that $\overline{\Omega} \subset \overline{D_1(0)} \setminus \{0\}$. By condition (5.2) we have

$$w^{\xi,\overline{\lambda}(\xi)}(\eta) - u(\eta) \leq 0 \qquad \text{for every } \eta \in \overline{\Omega} \text{ and every } \xi \in D_\varepsilon(0).$$

Recalling the notation $T = T(s, v, U)$ with $s > 0$, $v \in \mathbf{R}^{2n}$, $U \in \mathcal{S}^{2n\times 2n} \otimes J\mathbf{R}$, by the regularity of all the functions involved on $\overline{\Omega}$, for every $\xi \in D_\varepsilon(0)$ and every $\eta \in \overline{\Omega}$ we have

$$\begin{aligned}
0 \ &\leq \ T\big(u, \nabla_H u, \nabla_H^2 u\big) - T\big(w^{\xi,\overline{\lambda}(\xi)}, \nabla_H w^{\xi,\overline{\lambda}(\xi)}, \nabla_H^2 w^{\xi,\overline{\lambda}(\xi)}\big) \\
&= \int_0^1 \frac{d}{dr}\Big[T\big(ru + (1-r)w^{\xi,\overline{\lambda}(\xi)}, r\nabla_H u + (1-r)\nabla_H w^{\xi,\overline{\lambda}(\xi)}, \\
&\qquad\qquad\qquad\qquad r\nabla_H^2 u + (1-r)\nabla_H^2 w^{\xi,\overline{\lambda}(\xi)}\big)\Big]\, dr \\
&= c\big(w^{\xi,\overline{\lambda}(\xi)} - u\big) + \big\langle b, \nabla_H\big(w^{\xi,\overline{\lambda}(\xi)} - u\big)\big\rangle_{\mathbf{R}^{2n}} + \sum_{i,j=1}^{2n} a_{ij}\big[\nabla_H^2\big(w^{\xi,\overline{\lambda}(\xi)} - u\big)\big]_{ij} \\
&:= L\big(w^{\xi,\overline{\lambda}(\xi)} - u\big),
\end{aligned}$$

with $c \in \mathcal{C}^0(\overline{\Omega})$, $b = (b_1, \ldots, b_{2n}) \in \mathcal{C}^0(\overline{\Omega}, \mathbf{R}^{2n})$, $a_{ij} \in \mathcal{C}^0(\overline{\Omega})$ for every $i, j = 1, \ldots, 2n$ being defined by

$$\begin{aligned} c(\eta) &= -\int_0^1 \frac{\partial T}{\partial s}\big(ru(\eta) + (1-r)w^{\xi,\overline{\lambda}(\xi)}(\eta), r\nabla_H u(\eta) + (1-r)\nabla_H w^{\xi,\overline{\lambda}(\xi)}(\eta), \\ &\qquad\qquad r\nabla_H^2 u(\eta) + (1-r)\nabla_H^2 w^{\xi,\overline{\lambda}(\xi)}(\eta)\big)\, dr, \\ b_j(\eta) &= -\int_0^1 \frac{\partial T}{\partial v_j}\big(ru(\eta) + (1-r)w^{\xi,\overline{\lambda}(\xi)}(\eta), r\nabla_H u(\eta) + (1-r)\nabla_H w^{\xi,\overline{\lambda}(\xi)}(\eta), \\ &\qquad\qquad r\nabla_H^2 u(\eta) + (1-r)\nabla_H^2 w^{\xi,\overline{\lambda}(\xi)}(\eta)\big)\, dr, \\ a_{ij}(\eta) &= -\int_0^1 \frac{\partial T}{\partial U_{ij}}\big(ru(\eta) + (1-r)w^{\xi,\overline{\lambda}(\xi)}(\eta), r\nabla_H u(\eta) + (1-r)\nabla_H w^{\xi,\overline{\lambda}(\xi)}(\eta), \\ &\qquad\qquad r\nabla_H^2 u(\eta) + (1-r)\nabla_H^2 w^{\xi,\overline{\lambda}(\xi)}(\eta)\big)\, dr. \end{aligned}$$

By the regularity of u, w and since $\xi \circ \delta_{\overline{\lambda}(\xi)}(\eta) \in \overline{D_{\frac{3}{2}}(0)}$ for every $\xi \in D_\varepsilon(0)$ and every $\eta \in \overline{\Omega}$, we can find a, b, R_1, $R_2 \in \mathbf{R}^+$ with $b > a$ such that

$$\begin{aligned} &\big(ru + (1-r)w^{\xi,\overline{\lambda}(\xi)}, r\nabla_H u + (1-r)\nabla_H w^{\xi,\overline{\lambda}(\xi)}, r\nabla_H^2 u + (1-r)\nabla_H^2 w^{\xi,\overline{\lambda}(\xi)}\big) \\ &\qquad\qquad \in [a,b] \times B_{R_1}(0) \times B_{R_2}(0) \subset \mathbf{R}^+ \times \mathbf{R}^{2n} \times (\mathcal{S}^{2n\times 2n} \oplus J\mathbf{R}) \end{aligned}$$

for every $r \in [0,1]$, every $\xi \in D_\varepsilon(0)$ and every $\eta \in \overline{\Omega}$. Thus by our hypotheses on the operator T, see condition (22), the matrix $[a_{ij}(\eta)]_{i,j=1,\ldots,2n}$ is strictly positive definite on $\overline{\Omega}$, for every $\xi \in D_\varepsilon(0)$.

Since the vector fields $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ satisfy the Hörmander condition, see condition (24), by Theorem 4.1 in [25] we can conclude that for every $\xi \in D_1(0)$ we have either $\big(w^{\xi,\overline{\lambda}(\xi)} - u\big) \equiv 0$ or $\big(w^{\xi,\overline{\lambda}(\xi)} - u\big) < 0$ in $\overline{\Omega}$.

Now for any $m \in \mathbf{N}$ we can choose $\Omega = \Omega_m := D_1(0) \setminus D_{\frac{1}{m}}(0)$, and by condition (93) and the previous argument we can conclude that

$$\big(w^{\xi,\overline{\lambda}(\xi)} - u\big) < 0 \qquad \text{in } \overline{D_1(0)} \setminus D_{\frac{1}{m}}(0), \text{ for every } \xi \in D_\varepsilon(0) \text{ and every } m \in \mathbf{N}.$$

Thus our *claim* follows and (94) holds.

By the definition of $\overline{\lambda}(\xi)$ and by relation (94), then one has

$$\liminf_{\eta \to 0}[u(\eta) - w^{\xi,\overline{\lambda}(\xi)}(\eta)] = 0 \qquad \text{for every } \xi \in D_\varepsilon(0). \tag{95}$$

Thus in particular for every $\xi \in D_\varepsilon(0)$

$$w^{\xi,\overline{\lambda}(\xi)}(0) = \big(\overline{\lambda}(\xi)\big)^{\frac{Q-2}{2}} w(\xi) = \liminf_{\eta \to 0} u(\eta) := \alpha \in (0, \infty). \tag{96}$$

By conditions (94), (95) and (96) and since $\Delta_H u \leq 0$, Theorem 5.1 yields

$$V = \nabla_H w^{\xi,\overline{\lambda}(\xi)}(0) = \big(\overline{\lambda}(\xi)\big)^{\frac{Q}{2}} \nabla_H w(\xi) \qquad \text{for every } \xi \in D_\varepsilon(0), \tag{97}$$

for some fixed $V \in \mathbf{R}^{2n}$. Relations (96) and (97) in turn give

$$\nabla_H w(\xi) = \alpha^{-\frac{Q}{Q-2}} \big(w(\xi)\big)^{\frac{Q}{Q-2}} V \qquad \text{for every } \xi \in D_\varepsilon(0).$$

Thus for every $\xi \in D_\varepsilon(0)$ we get for $j = 1, \ldots, n$

$$\begin{aligned} X_j^2 w(\xi) &= \frac{Q}{Q-2}\alpha^{-\frac{2Q}{Q-2}}\big(w(\xi)\big)^{\frac{Q+2}{Q-2}} V_j^2, \\ Y_j^2 w(\xi) &= \frac{Q}{Q-2}\alpha^{-\frac{2Q}{Q-2}}\big(w(\xi)\big)^{\frac{Q+2}{Q-2}} V_{j+n}^2, \end{aligned}$$

and hence

$$\Delta_H w(\xi) = \frac{Q}{Q-2}\alpha^{-\frac{2Q}{Q-2}}\big(w(\xi)\big)^{\frac{Q+2}{Q-2}} |V|^2 \geq 0 \qquad \text{in } D_\varepsilon(0).$$

Then it follows that $\Delta_H(u-w) \leq 0$ in $D_\varepsilon(0) \setminus \{0\}$, with $u - w > 0$ in $D_\varepsilon(0) \setminus \{0\}$ by condition (i). Then Lemma 6.7 yields

$$\inf_{D_\varepsilon(0)\setminus\{0\}}(u-w) \geq \inf_{\partial D_\varepsilon(0)}(u-w) > 0,$$

and thus

$$\liminf_{\xi\to 0}[u(\xi) - w(\xi)] > 0,$$

which contradicts our initial assumption, condition (90). The proof of Theorem 1.7 is now complete. $\square$

6. **Appendix**

In this section we collect some technical results, which were used in the course of the proof of Theorem 1.3 in section 3.

Moreover we consider the CR map $\check\varphi = \varphi \circ \iota$, which we recall was defined in (29) by setting $\check\varphi(\xi) = \big(\check x, \check y, \check t\big)$ for every $\xi \in \mathbf{H}^n \setminus \{0\}$, with

$$\check x = -\frac{xt + y|z|^2}{|\xi|_H^4}, \quad \check y = \frac{yt - x|z|^2}{|\xi|_H^4}, \quad \check t = \frac{t}{|\xi|_H^4},$$

and for every $h = 1, \ldots, n$ we compute $\nabla_H(\check x_h)$, $\nabla_H(\check y_h)$, $\nabla_H(\check t)$, $\nabla_H^2(\check x_h)$, $\nabla_H^2(\check y_h)$, $\nabla_H^2(\check t)$.

Lemma 6.1. *For every $h, j = 1, \ldots, n$ we have*

$$\begin{aligned} X_j(\check x_h) = -Y_j(\check y_h) &= -\frac{t}{|\xi|_H^4}\,\delta_{jh} + \frac{1}{|\xi|_H^8}\Big(2\big(|z|^4 - t^2\big)(y_h x_j - x_h y_j) + 4t|z|^2(x_j x_h + y_j y_h)\Big) \\ X_j(\check y_h) = Y_j(\check x_h) &= -\frac{|z|^2}{|\xi|_H^4}\,\delta_{jh} + \frac{1}{|\xi|_H^8}\Big(2\big(|z|^4 - t^2\big)(x_j x_h + y_h y_j) + 4t|z|^2(y_j x_h - x_j y_h)\Big) \\ X_j(\check t) &= \frac{1}{|\xi|_H^8}\Big(2\big(|z|^4 - t^2\big)y_j - 4t|z|^2 x_j\Big) \\ Y_j(\check t\,) &= \frac{1}{|\xi|_H^8}\Big(2\big(t^2 - |z|^4\big)x_j - 4t|z|^2 y_j\Big) \\ T(\check x_h) &= \frac{1}{|\xi|_H^8}\Big(\big(t^2 - |z|^4\big)x_h + 2t|z|^2 y_h\Big) \\ T(\check y_h) &= \frac{1}{|\xi|_H^8}\Big(\big(|z|^4 - t^2\big)y_h + 2t|z|^2 x_h\Big) \\ T(\check t) &= \frac{1}{|\xi|_H^8}\big(|z|^4 - t^2\big) \end{aligned}$$

where δ_{jh} is the Kronecker's symbol.

Proof: One has just to use the definition of the vector fields T, X_j, Y_j for $j = 1, \dots, n$ given in section 2.1 in order to get the result. $\square$

Lemma 6.2. *For every $i, h, j = 1, \dots, n$ we have*

$$\begin{aligned}
X_jX_i(\check{x}_h) &= \frac{\delta_{ih}}{|\xi|_H^8}\big(2(t^2-|z|^4)y_j + 4t|z|^2x_j\big)\\
&\quad -\frac{8}{|\xi|_H^{12}}(|z|^2x_j + ty_j)\big(2(|z|^4-t^2)(x_iy_h - y_ix_h) + 4t|z|^2(x_ix_h + y_iy_h)\big)\\
&\quad +\frac{1}{|\xi|_H^8}\Big(8(|z|^2x_j - ty_j)(x_iy_h - y_ix_h) + 8(tx_j + |z|^2y_j)(x_ix_h + y_iy_h)\\
&\qquad +\delta_{ij}\big(2(|z|^4-t^2)y_h + 4t|z|^2x_h\big) + \delta_{jh}\big(2(t^2-|z|^4)y_i + 4t|z|^2x_i\big)\Big)
\end{aligned}$$

$$\begin{aligned}
X_jY_i(\check{x}_h) &= \frac{\delta_{ih}}{|\xi|_H^8}\big(2(|z|^4-t^2)x_j + 4t|z|^2y_j\big)\\
&\quad -\frac{8}{|\xi|_H^{12}}(|z|^2x_j + ty_j)\big(2(|z|^4-t^2)(y_iy_h + x_ix_h) + 4t|z|^2(y_ix_h - x_iy_h)\big)\\
&\quad +\frac{1}{|\xi|_H^8}\Big(8(|z|^2x_j - ty_j)(y_iy_h + x_ix_h) + 8(tx_j + |z|^2y_j)(y_ix_h - x_iy_h)\\
&\qquad +\delta_{ij}\big(2(|z|^4-t^2)x_h - 4t|z|^2y_h\big) + \delta_{jh}\big(2(|z|^4-t^2)x_i + 4t|z|^2y_i\big)\Big)
\end{aligned}$$

$$\begin{aligned}
Y_jX_i(\check{x}_h) &= \frac{\delta_{ih}}{|\xi|_H^8}\big(2(|z|^4-t^2)x_j + 4t|z|^2y_j\big)\\
&\quad +\frac{8}{|\xi|_H^{12}}(|z|^2y_j - tx_j)\big(2(|z|^4-t^2)(x_hy_i - x_iy_h) - 4t|z|^2(y_iy_h + x_ix_h)\big)\\
&\quad -\frac{1}{|\xi|_H^8}\Big(8(|z|^2y_j + tx_j)(y_ix_h - x_iy_h) + 8(|z|^2x_j - ty_j)(y_iy_h + x_ix_h)\\
&\qquad +\delta_{ij}\big(2(|z|^4-t^2)x_h - 4t|z|^2y_h\big) + \delta_{jh}\big(2(t^2-|z|^4)x_i - 4t|z|^2y_i\big)\Big)
\end{aligned}$$

$$\begin{aligned}
Y_jY_i(\check{x}_h) &= \frac{\delta_{ih}}{|\xi|_H^8}\big(2(|z|^4-t^2)y_j - 4t|z|^2x_j\big)\\
&\quad -\frac{8}{|\xi|_H^{12}}(|z|^2y_j - tx_j)\big(2(|z|^4-t^2)(y_iy_h + x_ix_h) + 4t|z|^2(y_ix_h - x_iy_h)\big)\\
&\quad +\frac{1}{|\xi|_H^8}\Big(8(|z|^2y_j + tx_j)(y_iy_h + x_ix_h) + 8(|z|^2x_j - ty_j)(x_iy_h - y_ix_h)\\
&\qquad +\delta_{ij}\big(2(|z|^4-t^2)y_h + 4t|z|^2x_h\big) + \delta_{jh}\big(2(|z|^4-t^2)y_i - 4t|z|^2x_i\big)\Big)
\end{aligned}$$

Moreover

$$\begin{aligned}
X_jX_i(\check{y}_h) &= X_jY_i(\check{x}_h) & \qquad Y_jY_i(\check{y}_h) &= -Y_jX_i(\check{x}_h),\\
Y_jX_i(\check{y}_h) &= Y_jY_i(\check{x}_h) & \qquad X_jY_i(\check{y}_h) &= -X_jX_i(\check{x}_h).
\end{aligned}$$

Finally

$$\begin{aligned} X_jX_i(\check{t}) &= -\frac{8}{|\xi|_H^{12}}(|z|^2x_j+ty_j)\big(2(|z|^4-t^2)y_i-4t|z|^2x_i\big) \\ &\quad +\frac{1}{|\xi|_H^8}\Big(8(|z|^2x_j-ty_j)y_i-8(tx_j+|z|^2y_j)x_i-4t|z|^2\delta_{ij}\Big) \\ X_jY_i(\check{t}) &= -\frac{8}{|\xi|_H^{12}}(|z|^2x_j+ty_j)\big(2(t^2-|z|^4)x_i-4t|z|^2y_i\big) \\ &\quad +\frac{1}{|\xi|_H^8}\Big(8(ty_j-|z|^2x_j)x_i-8(tx_j+|z|^2y_j)y_i+2(t^2-|z|^4)\delta_{ij}\Big) \\ Y_jY_i(\check{t}) &= -\frac{8}{|\xi|_H^{12}}(|z|^2y_j-tx_j)\big(2(t^2-|z|^4)x_i-4t|z|^2y_i\big) \\ &\quad +\frac{1}{|\xi|_H^8}\Big(8(|z|^2x_j-ty_j)y_i-8(|z|^2y_j+tx_j)x_i-4t|z|^2\delta_{ij}\Big). \end{aligned}$$

Proof: One needs only use the definition of the vector fields T, X_j, Y_j for $j=1,\dots,n$ and lemma 6.1 to conclude. $\square$

Remark 6.3. By Lemma 6.1 and Lemma 6.2, for every $h=1,\dots,n$ we have

$$\nabla_H(\check{x}_h)=-J\nabla_H(\check{y}_h),\qquad \nabla_H^2(\check{x}_h)=-J\nabla_H^2(\check{y}_h)$$

in $\mathbf{H}^n\setminus\{0\}$, where $J\in\mathrm{Mat}(2n,\mathbf{R})$ is the matrix defined in (16). Moreover

$$\begin{aligned} |\nabla_H(\check{x}_h)| &= \left(\sum_{j=1}^n\Big(|X_j(\check{x}_h)|^2+|Y_j(\check{x}_h)|^2\Big)\right)^{\frac12} = \frac{1}{|\xi|_H^2}, \\ |\nabla_H(\check{y}_h)| &= \left(\sum_{j=1}^n\Big(|X_j(\check{y}_h)|^2+|Y_j(\check{y}_h)|^2\Big)\right)^{\frac12} = \frac{1}{|\xi|_H^2} \end{aligned}$$

for every $\xi\in\mathbf{H}^n\setminus\{0\}$.

Lemma 6.4. *Let $v=(p,q)\in\mathbf{R}^n\times\mathbf{R}^n$, $v\neq0$, $s>0$ and let $\xi_0:=(x_0,y_0,t_0)$ with*

$$x_0=-\frac{(Q-2)s}{|v|^2}p,\quad y_0=-\frac{(Q-2)s}{|v|^2}q,\quad t_0=0.$$

Define $z_0=(x_0,y_0)$ and

$$\lambda:=|\xi_0|_H=\big(|x_0|^2+|y_0|^2\big)^{\frac12}=\frac{(Q-2)s}{|v|}.$$

Let

$$\psi(\xi)=\varphi\circ\iota\circ\delta_{\lambda^{-2}}(\xi)=\big(\lambda^2\check{x},\,\lambda^2\check{y},\,\lambda^4\check{t}\big)$$

for every every $\xi=(x,y,t)\in\mathbf{H}^n\setminus\{0\}$, and consider also a positive function $\phi\in\mathcal{C}^\infty(\mathbf{H}^n)$ such that $\phi(\xi_0)=s$, $\nabla_H\phi(\xi_0)=v$ and $\nabla_H^2\phi(\xi_0)=U$. Then one has $\phi_\psi(\psi^{-1}(\xi_0))=s$,

$\nabla_H \phi_\psi(\psi^{-1}(\xi_0)) = 0$ *and*

$$\nabla^2_H \phi_\psi(\psi^{-1}(\xi_0)) = G\Big[-\frac{Q}{Q-2}s^{-1}Jv\otimes Jv + \frac{2}{Q-2}s^{-1}v\otimes v + \frac{1}{Q-2}s^{-1}|v|^2 I_{2n} + J^T U J + \frac{4}{|v|^4}\Big(\langle v,\, Uv\rangle_{\mathbf{R}^{2n}}\, Jv\otimes Jv + \langle Jv,\, UJv\rangle_{\mathbf{R}^{2n}}\, v\otimes v - \langle Jv,\, Uv\rangle_{\mathbf{R}^{2n}}\, v\otimes Jv - \langle v,\, UJv\rangle_{\mathbf{R}^{2n}}\, Jv\otimes v\Big) + \frac{2}{|v|^2}\Big(Jv\otimes J^T U^T v - v\otimes J^T U^T Jv + J^T Uv\otimes Jv - J^T UJv\otimes v\Big)\Big] G, \tag{98}$$

with G, J being defined as in (16)*.*

Proof: Notice that $\psi = \check\varphi\circ\delta_{\lambda^{-2}}$, so that by formulae (23) and (29) and by Proposition 2.4, for every $\xi\in\mathbf{H}^n\setminus\{0\}$ one has

$$\begin{aligned} \left(\phi_{\check\varphi}\right)_{\delta_{\lambda^{-2}}}(\xi) &= \lambda^{-(Q-2)}\phi_{\check\varphi}(\lambda^{-2}x, \lambda^{-2}y, \lambda^{-4}t) \\ &= \left(\frac{\lambda}{|\xi|_H}\right)^{Q-2} \phi(\lambda^2\check x, \lambda^2\check y, \lambda^4\check t) \\ &= \phi_\psi(\xi). \end{aligned} \tag{99}$$

Now notice also that for every $\xi\in\mathbf{H}^n\setminus\{0\}$ one has $\psi^2(\xi)=\xi$, so that $\psi^{-1}(\xi)=\psi(\xi)$ for all $\xi\in\mathbf{H}^n\setminus\{0\}$. Then by the definition of λ and ξ_0 in particular one has

$$\psi^{-1}(\xi_0) \;=\; \psi(\xi_0) \;=\; (-y_0, -x_0, 0).$$

Thus using formula (99) one immediately gets

$$\phi_\psi(\psi^{-1}(\xi_0)) \;=\; \phi(\xi_0) \;=\; s$$

By formulae (99) and (26), for every $\xi=(x,y,t)\in\mathbf{H}^n\setminus\{0\}$ one also has

$$\nabla_H\phi_\psi(\xi) = \nabla_H\Big(\left(\phi_{\check\varphi}\right)_{\delta_{\lambda^{-2}}}\Big)(\xi) = \lambda^{-Q}\nabla_H\phi_{\check\varphi}(\lambda^{-2}x, \lambda^{-2}y, \lambda^{-4}t). \tag{100}$$

Evaluating equality (100) at $\psi^{-1}(\xi_0)$ and using formula (32) we get

$$\nabla_H\phi_\psi(\psi^{-1}(\xi_0)) = \frac{(Q-2)\lambda^{Q-2}}{|z_0|^Q}u(\xi_0)\begin{pmatrix} y_0 \\ x_0\end{pmatrix} + \frac{\lambda^Q}{|z_0|^Q}\,E\cdot\nabla_H u(\xi_0),$$

where the matrix E, which was defined in (33), is now evaluated at the point $\psi^{-1}(\xi_0)$. Recalling the definition of x_0 and y_0, from the previous equality one concludes that $\nabla_H\phi_\psi(\psi^{-1}(\xi_0)) = 0$ as claimed.

In a similar way, by formulae (27) one also has

$$\nabla^2_H\phi_\psi(\xi) = \nabla^2_H\Big(\left(\phi_{\check\varphi}\right)_{\delta_{\lambda^{-2}}}\Big)(\xi) = \lambda^{-(Q+2)}\nabla^2_H\phi_{\check\varphi}(\lambda^{-2}x, \lambda^{-2}y, \lambda^{-4}t), \tag{101}$$

so that

$$\nabla^2_H\phi_\psi(\psi^{-1}(\xi_0)) = \lambda^{-(Q+2)}\nabla^2_H\phi_{\check\varphi}\big(-\lambda^{-2}y_0, -\lambda^{-2}x_0, 0\big).$$

Now in order to get formula (98), and thus conclude the proof of the lemma, it's sufficient to substitute in the previous equality the expression for $\nabla^2_H\phi_{\check\varphi}$ which is provided by

formula (35), where all the functions appearing there are to be evaluated at the point $\xi = (-\lambda^{-2}y_0, -\lambda^{-2}x_0, 0)$. $\square$

Lemma 6.5. *Let $\lambda > 0$ and let*

$$\begin{aligned} \xi_0 &= (x_0, y_0, t_0) = (0, 0, \lambda^2), \\ \psi(\xi) &= \varphi \circ \iota \circ \delta_{\lambda^{-2}}(\xi) = \big(\lambda^2 \check{x},\, \lambda^2 \check{y},\, \lambda^4 \check{t}\big) \end{aligned}$$

for every every $\xi = (x, y, t) \in \mathbf{H}^n \setminus \{0\}$. Consider a positive function $\phi \in \mathcal{C}^\infty(\mathbf{H}^n)$ such that $\phi(\xi_0) = 1$, $\nabla_H \phi(\xi_0) = 0$ and $\nabla^2_H \phi(\xi_0) = U$. Then one has $\phi_\psi\big(\psi^{-1}(\xi_0)\big) = 1$, $\nabla_H \phi_\psi\big(\psi^{-1}(\xi_0)\big) = 0$ and

$$\nabla^2_H \phi_\psi\big(\psi^{-1}(\xi_0)\big) = -\frac{Q-2}{\lambda^2} J + GUG,$$

with G, J being defined as in (16).

Proof: One has $\psi(\xi_0) = \psi^{-1}(\xi_0) = \xi_0$ and $|\xi_0|_H = \lambda$. Thus, using formula (99) one immediately gets

$$\phi_\psi\big(\psi^{-1}(\xi_0)\big) = \phi(\xi_0) = 1.$$

Next, evaluating equality (100) at $\xi_0 = \psi^{-1}(\xi_0)$ and using formula (32) yields

$$\nabla_H \phi_\psi\big(\psi^{-1}(\xi_0)\big) = 0.$$

Finally, we evaluate (101) at the point $\xi_0 = \psi^{-1}(\xi_0)$ and we use formula (35). Since in this case we have $E = -G$, where E is the matrix defined in (33) now evaluated at the point $(0, 0, \lambda^{-2})$, we get

$$\nabla^2_H \phi_\psi\big(\psi^{-1}(\xi_0)\big) = -\frac{Q-2}{\lambda^2} J + GUG,$$

and the proof of the lemma is now complete. $\square$

One can prove the following lemma in the same way.

Lemma 6.6. *Let $\lambda > 0$ and let*

$$\begin{aligned} \xi_0 &= (x_0, y_0, t_0) = (0, 0, -\lambda^2), \\ \psi(\xi) &= \varphi \circ \iota \circ \delta_{\lambda^{-2}}(\xi) = \big(\lambda^2 \check{x},\, \lambda^2 \check{y},\, \lambda^4 \check{t}\big) \end{aligned}$$

for every every $\xi = (x, y, t) \in \mathbf{H}^n \setminus \{0\}$. Consider a positive function $\phi \in \mathcal{C}^\infty(\mathbf{H}^n)$ such that $\phi(\xi_0) = 1$, $\nabla_H \phi(\xi_0) = 0$ and $\nabla^2_H \phi(\xi_0) = U$. Then one has $\phi_\psi\big(\psi^{-1}(\xi_0)\big) = 1$, $\nabla_H \phi_\psi\big(\psi^{-1}(\xi_0)\big) = 0$ and

$$\nabla^2_H \phi_\psi\big(\psi^{-1}(\xi_0)\big) = \frac{Q-2}{\lambda^2} J + GUG,$$

with G, J being defined as in (16).

The following lemma is a consequence of Bony's Strong Maximum Principle, see [5].

Lemma 6.7. *Let $u \in \mathcal{C}^2\big(\overline{D_r(\xi_0)} \setminus \{\xi_0\}\big)$ for some $r > 0$ and $\xi_0 \in \mathbf{H}^n$. Assume*

i) $\Delta_H u \le 0$ *on* $D_r(\xi_0) \setminus \{\xi_0\}$,

ii) $u^-(\xi) = o\left(\frac{1}{|\xi - \xi_0|_H^{Q-2}}\right)$ *as* $\xi \to \xi_0$, *where* $u^- := -\min\{u, 0\}$.

Then $\inf_{D_r(\xi_0)\setminus\{\xi_0\}} u \geq \inf_{\partial D_r(\xi_0)} u.$

Proof: Up to a translation τ_{ξ_0} and a dilation δ_r, we can assume without loss of generality that $D_r(\xi_0) \equiv D_1(0)$.

Now consider the function $w(\xi) = \frac{1}{|\xi|_H^{Q-2}}$ on $\mathbf{H}^n \setminus \{0\}$, which is a multiple of the fundamental solution of $-\Delta_H$ on $\mathbf{H}^n$, centered at $0 \in \mathbf{H}^n$. Let $m_0 := \min_{\partial D_1(0)} u$ and define $v := \frac{m_0 - u}{w}$ on $\overline{D_1(0)} \setminus \{0\}$. Then for every $\varepsilon > 0$ one has

$$\Delta_H v + \frac{2}{w} \langle \nabla_H w, \nabla_H v \rangle_{\mathbf{R}^{2n}} = -\frac{\Delta_H u}{w} \geq 0$$

on $D_1(0) \setminus D_\varepsilon(0)$. Since $w \in C^\infty(\mathbf{H}^n \setminus \{0\})$ and $w > 0$ in $\mathbf{H}^n \setminus \{0\}$, by the Strong Maximum Principle proved by Bony in [5] one has that

$$\sup_{D_1(0)\setminus D_\varepsilon(0)} v \leq \sup_{\partial D_1(0)\cup \partial D_\varepsilon(0)} v^+ = \sup_{\partial D_\varepsilon(0)} v^+, \tag{102}$$

since by definition one has $v \leq 0$ on $\partial D_1(0)$. For every fixed $\xi \in D_1(0) \setminus \{0\}$, one has that $\xi \in D_1(0) \setminus D_\varepsilon(0)$ for every $\varepsilon > 0$ small enough. Then using (102) one obtains

$$v(\xi) \leq \sup_{\partial D_\varepsilon(0)} v^+ \leq \sup_{\partial D_\varepsilon(0)} \frac{c_0 + u^-}{w} \leq c_0 \varepsilon^{Q-2} + \sup_{\partial D_\varepsilon(0)} \frac{u^-}{w} \tag{103}$$

for every $\varepsilon > 0$ small enough. By condition *ii*), one has that $\frac{u^-(\eta)}{w(\eta)}$ tends to 0 as $\eta \to 0$, and hence $\sup_{\partial D_\varepsilon(0)} \frac{u^-}{w}$ also tends to 0 as $\varepsilon \to 0$. Then passing to the limit as $\varepsilon \to 0$ in (103) yields $v(\xi) \leq 0$, for every $\xi \in D_1(0) \setminus \{0\}$. By the definition of v thus we have

$$u(\xi) \geq c_0 = \min_{\partial D_1(0)} u \qquad \text{for every } \xi \in \overline{D_1(0)} \setminus \{0\}. \qquad \square$$

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